Chapter 3: Second order equations

Homogeneous equations \( y'' + p(t)y' + q(t)y = 0 \)
- Homogeneous equations with constant coefficient: \( ay'' + by' + cy = 0 \)
  - Characteristic equation: \( ar^2 + br + c = 0 \)
  - Real and distinct roots \( r_1 \) and \( r_2 \): \( y_1 = e^{r_1 t} \) and \( y_2 = e^{r_2 t} \) are the fundamental solutions
  - Complex roots: \( r_1 = \lambda + i\mu \), \( r_2 = \lambda - i\mu \); Euler’s formula: \( e^{it} = \cos t + i\sin t \); how to find real-valued solutions
  - Repeated roots: \( r_1 = r_2 \); fundamental solutions are \( y_1 = e^{r_1 t} \) and \( y_2 = te^{r_1 t} \)
- Reduction of order
  - One solution \( y_1(t) \) is known for \( y'' + p(t)y' + q(t)y = 0 \)
  - Find a second solution by setting \( y(t) = v(t)y_1(t) \) for some function \( v(t) \)
  - Plug \( y(t) \) into the differential equation to find an appropriate \( v(t) \)

Nonhomogeneous equations \( y'' + p(t)y' + q(t)y = g(t) \)
- General solutions are of the form \( y(t) = c_1y_1(t) + c_2y_2(t) + Y(t) \), where \( y_1 \) and \( y_2 \) are fundamental solutions for the corresponding homogeneous equation, and \( Y(t) \) is a particular solution of the nonhomogeneous equation
- Method of undetermined coefficients
  - Used to solve \( ay'' + by' + cy = g(t) \), where \( g(t) \) is a “simple” function – exponentials, polynomials, sines and cosines
  - If \( g(t) = (a_0 + a_1 t + \cdots + a_n t^n) e^{\alpha t} \) \( \begin{cases} \sin \beta t \\ \cos \beta t \end{cases} \),
  then assume \( Y(t) \) has the form
  \[
  Y(t) = [(A_0 + A_1 t + \cdots + A_n t^n) e^{\alpha t} \sin \beta t \\
  + (B_0 + B_1 t + \cdots + B_n t^n) e^{\alpha t} \cos \beta t] t^s,
  \]
  where \( s \) is the number of times \( \alpha + \beta i \) is a root of the characteristic equation
  - Plug \( Y(t) \) into the differential equation to determine coefficients
- Variation of parameters
  - A more general method for solving \( y'' + p(t)y' + q(t)y = g(t) \)
  - The nonhomogeneous solution is \( Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \), where \( y_1 \) and \( y_2 \) are fundamental solutions of the homogeneous equation and \( u_1 \) and \( u_2 \) are defined as
  \[
  u_1(t) = - \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \, dt \quad \text{and} \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \, dt
  \]
  (This was achieved by assuming that \( u_1'y_1 + u_2'y_2 = 0 \) and also by plugging \( Y(t) \) into the differential equation to come up with a second condition on \( u_1(t) \) and \( u_2(t) \))
Chapter 5: Power Series Solutions

- Modifying a power series (shifting index, rewrite with \(x^n\) terms, etc), radius of convergence, interval of convergence
- \(P(x)y'' + Q(x)y' + R(x)y = 0\), where \(P(x), Q(x),\) and \(R(x)\) are polynomials
- A power series solution has the form \(y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n\) about a point \(x_0\) such that \(P(x_0) \neq 0\) (ordinary point)
- A recurrence relation for the coefficients \(a_n\) is found by plugging in \(y(x)\) into the differential equation and matching up powers of \(x\)