1. Find the eigenvalues and eigenvectors of the following matrices. What are the dimensions of the eigenspaces? If possible, diagonalize the matrices (find $P$ and $D$ such that $M = PDP^{-1}$; you don’t need to compute $P^{-1}$).

\[
A = \begin{pmatrix}
-3 & 0 & 0 \\
2 & -1 & -2 \\
-2 & -2 & -1
\end{pmatrix}
\quad B = \begin{pmatrix}
-2 & 0 & 0 \\
1 & -1 & -1 \\
-1 & -1 & -1
\end{pmatrix}
\quad C = \begin{pmatrix}
-2 & 0 & -1 \\
0 & 2 & 0 \\
-1 & 0 & -2
\end{pmatrix}
\]

\[E = \begin{pmatrix}
6 & 3 & -8 \\
0 & -2 & 0 \\
1 & 0 & -3
\end{pmatrix}
\quad F = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 2 & 0 \\
0 & 1 & -3
\end{pmatrix}
\]

2. Consider an $m \times n$ matrix $A$. What are the fundamental subspaces? How are the subspaces and their orthogonal complements related?

3. Consider the matrix

\[
A = \begin{pmatrix}
1 & -1 & 0 \\
3 & 1 & -4 \\
0 & -2 & 2 \\
1 & 3 & -3
\end{pmatrix}
\]

Find a basis for the orthogonal complement of $\text{col}(A)$.

4. The matrix $A$ is row equivalent to the matrix $B$.

\[
A = \begin{pmatrix}
1 & 3 & 0 & -1 & 0 \\
-1 & -3 & 1 & -4 & 2 \\
0 & 0 & -1 & 5 & -5 \\
2 & 6 & 0 & -2 & 0
\end{pmatrix}
\quad B = \begin{pmatrix}
1 & 3 & 0 & -1 & 0 \\
0 & 0 & 1 & -5 & 2 \\
0 & 0 & 0 & 0 & -3 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Find a basis for the orthogonal complement of $\text{row}(A)$.
5. Let \( W = \text{span}\{w_1, w_2\} \) be a subspace in \( \mathbb{R}^n \) with \( \{w_1, w_2\} \) an orthogonal basis for \( W \). Let \( y, z \in \mathbb{R}^n \) and with \( z = y - \frac{y \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{y \cdot w_2}{w_2 \cdot w_2} w_2 \). Show that \( z \) is in \( W^\perp \).

6. Consider the following vectors and subspaces.
   
   \( \circ \ V = \text{span}\{v_1, v_2\} \), where
   \[
   v_1 = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 5 \\ -5 \\ -1 \\ 2 \end{pmatrix}
   \]
   
   \( \circ \ W = \text{span}\{w_1, w_2\} \), where
   \[
   w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } w_2 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}
   \]
   
   (a) Are \( v_1 \) and \( v_2 \) orthogonal? Are \( w_1 \) and \( w_2 \) orthogonal?
   
   (b) Use the Gram-Schmidt process to find an orthogonal basis for each of the subspaces.
   
   (c) Give an orthonormal basis for each of the subspaces.
   
   (d) Let \( x = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} \). Compute the projection of \( x \) onto \( W \). What is the distance from \( x \) to the subspace \( W \)?
   
   (e) If a vector \( x \) is in the span of \( w_1 \) and \( w_2 \), what is the projection of \( x \) onto \( W \)?

7. Consider an \( m \times n \) matrix \( A \). (See Chapter 6, Theorem 3 and class notes)
   
   \( \circ \) Suppose a vector \( x \) is in \( \text{null}(A) \). Show that \( x \) is orthogonal to the rows of the matrix \( A \).
   
   \( \circ \) Suppose a vector \( y \) is in \( \text{row}(A)^\perp \). Show that \( Ay = 0 \).

8. Suppose \( U \) is an orthonormal matrix (has orthonormal columns). Prove the following properties:
   
   \( \circ \) \( \|Ux\| = \|x\| \)
   
   \( \circ \) \( (Ux) \cdot (Uy) = x \cdot y \)

9. Consider the following matrix and vector
   \[
   A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}
   \]
   
   \( \circ \) Find the projection of \( b \) onto \( \text{col}(A) \).
   
   \( \circ \) Find the least-squares solution of \( Ax = b \).

10. Consider the following matrix and vector
    \[
    A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}
    \]
    
    \( \circ \) Is the system \( Ax = b \) consistent?
Find the projection of $b$ onto $\text{col}(A)$.

Find the least-squares solution of $Ax = b$.

11. Let $A$ be an $n \times n$ matrix. What are some statements that are equivalent to the statement that $A$ is invertible? (Invertible matrix theorem)

12. Let $A$ be the following matrix:

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$ 

Find an orthonormal basis for $\text{col}(A)$ using the Gram-Schmidt process.

13. Consider the following vectors:

$$u_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}, \quad \text{and} \quad y = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$ 

Let the subspace $W = \text{span}\{w_1, w_2\}$. Write $y$ as the sum of a vector in $W$ and a vector in $W^\perp$. 