Instructions:

1. Write your Name, Section, and PID on the front of your blue book.

2. Write the Version of your exam on the front of your blue book.

3. Read each question carefully. Justify your answers. No credit will be given for unsupported answers.

4. Write your solutions clearly:
   (a) Carefully indicate the number and letter of the question you are answering.
   (b) Start each question on a new side of a page.

5. No electronic devices are allowed.

6. You may use your handwritten (8.5”x11”) sheet of notes, but nothing else during this exam.
0. (2 pts) Read and follow the instructions on the other side of this page.

1. (12 pts) Consider the following matrix:

\[ A = \begin{bmatrix}
4 & 1 & 0 \\
2 & -1 & 3 \\
0 & 5 & -10 \\
2 & 0 & 1
\end{bmatrix}. \]

(a) (4 pts) Is \( A\vec{x} = \vec{b} \) consistent for all \( \vec{b} \in \mathbb{R}^4 \)?

This is equivalent to asking for every vector \( \vec{b} \), does \( A\vec{x} = \vec{b} \) have a solution? By Theorem 4 in Chapter 1, this is equivalent to checking if \( A \) has a pivot in every row.

\[
\begin{align*}
A &= \begin{bmatrix}
4 & 1 & 0 \\
2 & -1 & 3 \\
0 & 5 & -10 \\
2 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
4 & 1 & 0 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \\
&\rightarrow \begin{bmatrix}
4 & 1 & 0 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \\
&\rightarrow \begin{bmatrix}
4 & 1 & 0 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

Since \( A \) does not have a pivot in every row, \( A\vec{x} = \vec{b} \) is not consistent for all \( \vec{b} \).

(b) (5 pts) Find the solution set of the homogeneous equation defined by \( A \).

\[
\begin{bmatrix}
4 & 1 & 0 & 0 \\
2 & -1 & 3 & 0 \\
0 & 5 & -10 & 0 \\
2 & 0 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
4 & 1 & 0 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
4 & 1 & 0 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
4 & 0 & 2 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

We see that \( x_3 \) is a free variable and \( x_1 = -\frac{1}{2}x_3 \) and \( x_2 = 2x_3 \). Therefore, the solutions of \( A\vec{x} = \vec{0} \) are

\[
\vec{x} = x_3 \begin{bmatrix}
-\frac{1}{2} \\
2 \\
1
\end{bmatrix} = \text{span} \left\{ \begin{bmatrix}
-\frac{1}{2} \\
2 \\
1
\end{bmatrix} \right\}
\]
(c) (3 pts) Are the columns of \( A \) linearly independent or dependent? If dependent, find a linear dependence relation.

The columns of \( A \) are linearly independent if and only if \( A\vec{x} = \vec{0} \) has only the trivial solution. From part (b), the homogeneous equation has many solutions. Therefore, the columns are dependent.

One linear dependence relation is

\[
\begin{bmatrix}
-1 \\
2 \\
2
\end{bmatrix} + 4 \begin{bmatrix}
4 \\
2 \\
0
\end{bmatrix} + 2 \begin{bmatrix}
1 \\
-1 \\
0
\end{bmatrix} = 0
\]

(when \( x_3 = 2 \) from part (b))

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2. (10 pts) Consider the transformation \( T(\vec{x}) = A\vec{x} \) and the vector \( \vec{b} \), where

\[
A = \begin{bmatrix}
1 & -1 & 0 & 3 \\
0 & 3 & 0 & -1 \\
0 & 0 & 0 & -2
\end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix}
1 \\
7 \\
-4
\end{bmatrix}.
\]

(a) (2 pts) What is the domain and codomain of \( T \)?

Since \( A \) is \( 3 \times 4 \), the domain of \( T \) is \( \mathbb{R}^4 \) and the codomain is \( \mathbb{R}^3 \).

(b) (4 pts) Is \( T(\vec{x}) \) one-to-one?

\( T(\vec{x}) \) is one-to-one if and only if the columns of \( A \) are linearly independent. In this problem, \( A \) has a zero vector as a column. Therefore the columns are linearly dependent and \( T(\vec{x}) \) is not one-to-one.

(c) (4 pts) Is \( \vec{b} \) in the range of \( T \)?

\( A \) is in row echelon form and has a pivot in every row. Therefore, \( T(\vec{x}) \) is onto and \( A\vec{x} = \vec{b} \) has a solution for all vectors \( \vec{b} \). This means that \( \vec{b} \) must be in the range of \( T \).

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3. (4 pts) Consider the following vectors:

\[
\vec{v}_1 = \begin{bmatrix}
-1 \\
0
\end{bmatrix}; \quad \vec{v}_2 = \begin{bmatrix}
1 \\
3
\end{bmatrix}
\]

(a) (2 pts) Find two vectors (not \( \vec{v}_1 \) or \( \vec{v}_2 \)) that are in the span of \( \{ \vec{v}_1, \vec{v}_2 \} \).

(b) (2 pts) Suppose \( T \) is a linear transformation such that:

\[
T(\vec{x}) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ with } T(\vec{v}_1) = \vec{y} \text{ and } T(\vec{v}_2) = \vec{z}.
\]

What does \( T \) map \( 2\vec{v}_1 + \vec{v}_2 \) to?
Since $T$ is a linear transformation,
\[ T(2\vec{v}_1 + \vec{v}_2) = T(2\vec{v}_1) + T(\vec{v}_2) = 2T(\vec{v}_1) + T(\vec{v}_2) = 2\vec{y} + \vec{z}. \]

4. (4 pts) Consider the transformation $T(\vec{x}) : \mathbb{R}^2 \to \mathbb{R}^3$ defined as:
\[ T(\vec{x}) = \begin{bmatrix} 2x_1 - 5x_2 \\ -x_1 + 2x_2 \\ x_2 \end{bmatrix} \]
Find the standard matrix of this transformation.

\[ T(\vec{x}) = \begin{bmatrix} 2 & -5 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \to A = \begin{bmatrix} 2 & -5 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \]

5. (4 pts) Let $A$ be a $3 \times 4$ matrix, $B$ be a $4 \times 2$ matrix, and $C$ be a $4 \times 4$ matrix. Provide the dimensions of the following matrices or state that the matrix is not defined.
(a) $AB$  (b) $B^TC$  (c) $A^TC$  (d) $B^2$

(a) $AB$ is $3 \times 2$  (b) $B^TC$ is $2 \times 4$  (c) $A^TC$ is not defined  (d) $B^2$ is not defined