1. State whether the following matrices are invertible. If invertible, find the inverse matrix.

   (a) \( A = \begin{pmatrix} -1 & 7 \\ 2 & 3 \end{pmatrix} \)  
   (b) \( B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \)  
   (c) \( C = \begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix} \)  
   (d) \( D = \begin{pmatrix} 0 & -2 \\ -4 & 0 \end{pmatrix} \)  
   (e) \( M = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \)  
   (f) \( BD \), where \( B \) and \( D \) are defined above

   For a 2 \( \times \) 2 matrix \( M \), if \( \det(M) \neq 0 \), then we have an explicit formula for the inverse:

   \[
   M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
   \]

   For part (e), we know that generally, in order to find the inverse of a matrix, we have to try to reduce it to the identity matrix

   \[
   \begin{bmatrix} A & -I \\ \end{bmatrix} \sim \begin{bmatrix} I & A^{-1} \end{bmatrix}
   \]

   If we can’t, then \( A \) is not invertible.

   For part (f), we use the fact that the inverse of a product of matrices is the product of the inverses in reverse order. The inverse of \( BD \) is \( (BD)^{-1} = D^{-1}B^{-1} \).

2. Suppose \( T(\vec{x}) = A\vec{x} \) is not one-to-one, where \( A \) is an \( m \times n \) matrix. Then there exists a vector \( \vec{b} \in \mathbb{R}^m \) such that \( \vec{b} = T(\vec{x}) = T(\vec{y}) \), where \( \vec{x} \neq \vec{y} \) with \( \vec{x} \) and \( \vec{y} \) in \( \mathbb{R}^n \). Use this information to show that there exists a nonzero vector in the nullspace of \( A \).

   (This proves one direction of the statement that \( T(\vec{x}) \) is one-to-one if and only if \( \text{null}(A) = \{0\} \)).

   If \( \vec{b} = T(\vec{x}) = T(\vec{y}) \), then we have \( A\vec{x} = A\vec{y} \). Then \( A\vec{x} - A\vec{y} = A(\vec{x} - \vec{y}) = 0 \) so that the vector \( \vec{x} - \vec{y} \) is in the nullspace of \( A \). But \( \vec{x} \neq \vec{y} \), therefore, we have a nonzero vector \( \vec{x} - \vec{y} \) in \( \text{null}(A) \).

3. Suppose \( A \) and \( B \) are \( n \times n \) and invertible.

   (a) What is the inverse of \( AB \)?
(b) What is the inverse of $AB^T$?
(c) What is the inverse of $A^2B$?
(d) What is the inverse of $(AB^2)^T$?

Assuming that all the matrices involved are invertible:

- The inverse of a product of matrices is the product of the inverses in reverse order.
- The transpose of a product of matrices is the product of the transposes in reverse order.
- The inverse of the transpose is the transpose of the inverse.

(a) $(AB)^{-1} = B^{-1}A^{-1}$
(b) $(AB^T)^{-1} = (B^T)^{-1}A^{-1} = (B^{-1})^TA^{-1}$
(c) $(A^2B)^{-1} = B^{-1}A^{-1}A^{-1}$
(d) $(AB^2)^T = (AB)B = B^TB^TA^T$. Then the inverse is $((AB^2)^T)^{-1} = (B^T)^{-1}(B^{-1})^T(B^{-1})^T = (A^T)^{-1}(B^T)^{-1}(B^{-1})^T(B^{-1})^T$

4. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 3 & 0 & 1 \\ -1 & 1 & -2 & 3 \\ 0 & -4 & 2 & -3 \end{pmatrix}.$$  

(a) For what value of $k$ is $\text{col}(A)$ a subspace of $\mathbb{R}^k$?

Since $A$ is a $3 \times 4$ matrix, $\text{col}(A)$ is a subspace of $\mathbb{R}^3$.

(b) Find a basis for $\text{col}(A)$.

$A$ is row equivalent to

$$\begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 4 & -2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

The pivots are in columns 1, 2, and 4. Thus the set of vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} \right\}$$

forms a basis for the columnspace of $A$.

(c) What is the dimension of $\text{col}(A)$?

$$\text{dim} \text{col}(A) = 3$$

(d) For what value of $k$ is $\text{null}(A)$ a subspace of $\mathbb{R}^k$?

Since $A$ is a $3 \times 4$ matrix, $\text{null}(A)$ is a subspace of $\mathbb{R}^4$. 

2
(e) Find a basis for null(A).

\[
\begin{bmatrix} A & 0 \end{bmatrix} \text{ is row equivalent to}
\begin{bmatrix} 1 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.
\]

Solutions to \( A\vec{x} = 0 \) have the form:

\[
\vec{x} = x_3 \begin{pmatrix} -\frac{3}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}.
\]

The basis for the nullspace is

\[
\left\{ \begin{pmatrix} -\frac{3}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \right\}.
\]

(f) What is the dimension of null(A)?

\[ \dim \text{null}(A) = 1 \]

(g) What is the rank of \( A \) and \( A^T \)?

Remember that the rank of a matrix is equal to the dimension of its columnspace.

5. Consider the following matrices:

\[
A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}
\]

(a) What is the rank of \( A \)? Does \( A\vec{x} = b \) have a solution for all \( b \)?

\[ \text{rank}(A) = 3 \text{ and } A \text{ is a } 3 \times 4 \text{ matrix. Since } \text{rank}(A) = 3, \text{ col}(A) = \mathbb{R}^3 \text{ so } A\vec{x} = b \text{ has a solution for all vectors } \vec{b}. \]

(b) What is the rank of \( B \)? Does \( B\vec{x} = b \) have a solution for all \( b \)?

\[ \text{rank}(B) = 3 \text{ and } B \text{ is a } 4 \times 3 \text{ matrix. Since } \text{rank}(A) \neq 4, \text{ col}(A) \neq \mathbb{R}^4 \text{ so } B\vec{x} = b \text{ does not have a solution for all vectors } \vec{b}. \]
6. The matrix $A$ is row equivalent to the matrix $B$.

$$A = \begin{pmatrix} 1 & 3 & 0 & -1 & 0 \\ -1 & -3 & 1 & -4 & 2 \\ 0 & 0 & -1 & 5 & -5 \\ 2 & 6 & 0 & -2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

(a) What is the rank of $A$ and $A^T$?
(b) What is the dimension of $\text{col}(A)$?
(c) What is the dimension of $\text{null}(A)$?
(d) Find a basis for $\text{col}(A)$.
(e) Find a basis for $\text{row}(A)$.

The basis for $\text{row}(A)$ is
\begin{align*}
\begin{pmatrix} 1 \\ 3 \\ 0 \\ -1 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 1 \\ -5 \\ 2 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ -3 \end{pmatrix}
\end{align*}

(f) Find a basis for $\text{null}(A)$.
(g) Find a basis for $\text{col}(A^T)$.

Recall that $\text{row}(A) = \text{col}(A^T)$.

(h) Find a basis for $\text{row}(A^T)$.

Recall that $\text{row}(A^T) = \text{col}(A)$.

7. Let $V$ be a 3-dimensional vector space.

(a) Suppose the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent. Do the vectors form a basis for $V$? Why or why not?
(b) Suppose the vectors $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ span $V$. Do the vectors form a basis for $V$? Why or why not?

Recall the Basis Theorem (from Section 4.5).

8. Let $W = \left\{ \begin{pmatrix} -3a + c \\ b \\ -2b + 3c \\ a \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$

(a) Show $W$ is a subspace of $\mathbb{R}^4$. 

\[
\begin{bmatrix}
-3a + c \\
\frac{b}{c} \\
-2b + 3c \\
a
\end{bmatrix}
= a \begin{pmatrix}
-3 \\
0 \\
0 \\
1
\end{pmatrix}
+ b \begin{pmatrix}
0 \\
1 \\
-2 \\
0
\end{pmatrix}
+ c \begin{pmatrix}
0 \\
0 \\
3 \\
0
\end{pmatrix}
\]

So \(W\) is equal to
\[
\text{span}\left\{ \begin{pmatrix}
-3 \\
0 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
-2 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
3 \\
0
\end{pmatrix} \right\}.
\]

The span of any set of vectors in \(\mathbb{R}^4\) is a subspace. Thus, \(W\) is a subspace of \(\mathbb{R}^4\).

(b) What is the dimension of \(W\)?

The dimension of \(W\) is 3.

(c) Find a matrix \(A\) such that \(\text{col}(A) = W\).

From part (a), since \(W\) is equal to
\[
\text{span}\left\{ \begin{pmatrix}
-3 \\
0 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
-2 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
3 \\
0
\end{pmatrix} \right\},
\]

\(W\) is also equal to the column space of \(A\)

\[
A = \begin{pmatrix}
-3 & 0 & 1 \\
0 & 1 & 0 \\
0 & -2 & 3 \\
1 & 0 & 0
\end{pmatrix}.
\]

9. Let \(W = \left\{ \begin{pmatrix}
r \\
s
\end{pmatrix} : r, s \in \mathbb{R} \text{ and } r + s = 1 \right\}\). Is \(W\) a subspace of \(\mathbb{R}^2\)?

Remember a subspace has to satisfy 3 conditions. It must contain the zero vector, and it must be closed under addition and scalar multiplication.

Notice that the zero vector \(\begin{pmatrix}0 \\ 0\end{pmatrix}\) is not in \(W\) because \(0 + 0 \neq 1\). Thus \(W\) is not a subspace.

10. Suppose \(A\) is a \(8 \times 6\) matrix with \(\text{rank}(A) = 5\). Is it possible to find a vector \(b\) such that \(Ax = b\) is inconsistent?

Since \(\text{rank}(A) = 5 < 8 = m\), \(\text{col}(A)\) is not all of \(\mathbb{R}^8\). Therefore, it is possible to find \(\vec{b}\) such that \(Ax = \vec{b}\) is inconsistent.
11. Suppose $A$ is $10 \times 12$ and $\dim \text{null}(A) = 2$. Is there a vector $b$ such that $A\vec{x} = b$ is inconsistent?

We use the Rank Theorem: $n = \text{rank}(A) + \dim \text{null}(A)$.
For this problem, $n = 12$. Therefore, $\text{rank}(A) = 12 - 2 = 10$. So we have $\text{rank}(A) = m = 10$.
Thus, $\text{col}(A) = \mathbb{R}^m$ and $A\vec{x} = \vec{b}$ is consistent for all vectors $\vec{b}$.

12. Suppose $A$ is $10 \times 12$ and there is a solution for some nonhomogeneous system with 3 free variables. Can we find a vector $b$ such that $A\vec{x} = b$ is inconsistent?

Three free variables means $\dim \text{null}(A) = 3$. Then by the Rank Theorem, $\text{rank}(A) = 12 - \dim \text{null}(A) = 9 < 10$. $\text{col}(A)$ is not all of $\mathbb{R}^{10}$. Therefore, it is possible to find $\vec{b}$ such that $A\vec{x} = \vec{b}$ is inconsistent.

13. $A$ is $8 \times 11$. Is it possible for the solutions of $A\vec{x} = 0$ to be multiples of one vector?

$\dim \text{null}(A) = 1$. Then by the Rank Theorem, $\text{rank}(A) = 11 - \dim \text{null}(A) = 10$. This isn’t possible though because the rank of a matrix is equal to the number of pivots. $A$ has only 8 rows, so $\text{rank}(A) \neq 10$.

14. Suppose $A$ is $n \times n$. Provide three equivalent statements to the statement that $A$ is invertible.

15. Let $\mathcal{B}$ be a basis where

$$\mathcal{B} = \{b_1, b_2\}, \quad b_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

(a) Let $\vec{x} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$. Find the coordinate vector $[\vec{x}]_\mathcal{B}$ of the vector $\vec{x}$ relative to the basis $\mathcal{B}$.

(b) What is the change-of-coordinate matrix from $\mathcal{B}$ to the standard basis?