General Information:
- Wednesday, March 16 at 8-11 am in our classroom Peter 110.
- covers Chapter 1 (1.1-1.9), Chapter 2 (2.1-2.3, 2.5), Chapter 3 (3.1-3.3),
  Chapter 4 (4.1-4.7), Chapter 5 (3.1-3.3), Chapter 6 (6.1-6.5)
- will be cumulative, but emphasizing Chapters 5 and 6 (about half the problems)
- 12 problems
- closed note, closed book, no calculators
- Justify your answers and show all your work! The grader reserves the right to take
  off points if insufficient work/justification of answer is shown.
- A good way to study for the final is to go through the homework problems and the
  second midterm. It will be similar in style to the second midterm, and contain a
  mixture of easy, medium and a few harder problems.

Guide for the Material

- Vector Spaces
  - Know what a vector space is.
  - Know what a subspace is, and how to show that something is a subspace, and
    how to show something isn’t a subspace.
  - Basis
    * Know the definition and the meaning of a basis!!! A set S for a vector space
      V is a basis if it is linearly independent and span(S)=V.
    * Consider a finite-dimensional vector space V.
      - (Spanning Theorem) Know that any spanning set in V can be reduced to
        a basis by removing “redundant” vectors that are a linear combination of
        the rest of the vectors.
      - (Extension Theorem) Know that any linearly independing set in V can
        be extended to a basis.
      - (Dimension Theorem) Know that every basis for V has the same amount
        of elements, and this number is called dim(V), or the dimension of V.
    * (Unique Representation Theorem) Know that each element in V can be
      written only uniquely as a linear combination of vectors in the basis.
  - Isomorphism
    * Let dim(V)=n. Know the coordinate mapping $T : V \to \mathbb{R}^n$ defined by
      $T(x) = [x]_\beta$, and that it is onto and one-to-one. Therefore, there is an
      isomorphism from any n-dimensional vector space V to $\mathbb{R}^n$! So every
      n-dimensional vector space behaves like $\mathbb{R}^n$. 

Therefore, to answer some questions, one can use vectors in \( \mathbb{R}^n \) instead of working with the original vector space. For example, to write \( 1 + 4t + 7t^2 \) as a linear combination of the basis \( 1 + t^2, t + t^2, 1 + 2t + t^2 \), you can work with the augmented system \[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 2 & 4 \\
1 & 1 & 1 & 7
\end{bmatrix}.
\] The solution to this system will give the coefficients of the linear combination.

• Linear Transformations

- Know that we can define transformations in a more general way: \( T: V \rightarrow W \), where \( V \) and \( W \) are any vector spaces. An example is \( T: P_2 \rightarrow M_{2\times2} \) defined by \( T(a + bx + cx^2) = \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} \).
- We will focus on linear transformations.
- Know the definition of the kernel (also called nullspace) of a linear transformation \( T: V \rightarrow W \). This is just a set of vectors \( x \) that satisfy \( T(x) = 0 \), where \( 0 \) indicates the general zero vector for the vector space \( W \).
- Know the definition of the range of a linear transformation \( T: V \rightarrow W \). The range will be the set of vectors \( T(x) \) where \( x \in V \).
- Know how to find the kernel and range of a linear transformation \( T: V \rightarrow W \).
- Know what it means for a transformation to be one-to-one. A linear transformation \( T \) is one-to-one if the only solution to \( T(x) = 0 \) is \( x = 0 \).
- Know what it means for a transformation to be onto. A transformation \( T \) is onto if for every \( y \in W \), there exists an \( x \) such that \( T(x) = y \).
- Linear transformations from \( \mathbb{R}^n \rightarrow \mathbb{R}^m \):
  * Any linear transformation \( \mathbb{R}^n \rightarrow \mathbb{R}^m \) can be written as \( T(x) = Ax \).
  * The matrix \( A \) is called the standard matrix: know how to calculate it.
  * The range of such a linear transformation \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is just the column space of \( A \), the kernel or nullspace of \( T \) is just the nullspace of \( A \).
  * If \( n = m \), linear \( T: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is invertible if and only if \( A \) is invertible. Recall that a transformation \( T \) from \( T: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is invertible if there exists a transformation \( S: \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( T(S(x)) = S(T(x)) \).
  * Recall that for a linear transformation \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) written as \( T(x) = Ax \), to check whether or not \( T \) is onto, check if each row of \( \text{rref}(A) \) has a pivot. If yes, it is onto. If not, it is not onto.
  * Recall that for a linear transformation \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) written as \( T(x) = Ax \), to check whether or not \( T \) is one-to-one, check if the columns of \( A \) are linearly independent. If yes, it is one-to-one. Otherwise, it is not. Also, if \( T(x) = 0 \) only has the trivial solution, then \( T \) is one-to-one. Otherwise, it is not.

• Matrices

- Know how to find if an \( n \times n \) matrix is invertible, and if it is, what the inverse is. Remember that one can use the Invertible Matrix Theorem.
Know how to calculate the nullspace, rowspace and column space and their bases.

- **Rank Theorem**
  - (Rank Theorem) Important!!! Says that $\dim(\text{col}(A)) + \dim(\text{nul}(A)) = n$, for an $m \times n$ matrix $A$.
  - Let $T : V \to W$ be a linear transformation from vector spaces $V$ to $W$, of dimension $n$ and $m$, respectively. Then, a variant of the rank theorem for transformations is: $\dim(\text{range}(T)) + \dim(\text{kernel}(T)) = n$.

- **Inverse Matrix Theorem**
  - This theorem applies only to $n \times n$ matrices.
  - For a $n \times n$ matrix $A$, the following conditions are either ALL true or ALL false:
    - $A$ is invertible.
    - $T(x) = Ax$ is one-to-one.
    - $T(x) = Ax$ is onto.
    - $\det(A) \neq 0$.

- **Determinants**
  - Know how to compute the determinant of any $n \times n$ matrix.
  - However, keep in mind the following shortcuts!!!
    - If a matrix has a lot of zeros across a row or column, use that row or column in the cofactor expansion to find the det!
    - The determinant of a triangular matrix is the product of its diagonal entries!!!
    - Know how the three basic row operations change the determinant.
    - $\det(A) = 0$, if certain columns or rows of $A$ are linearly dependent!!!
    - The determinant of a matrix containing columns or rows of all zeros is 0.
  - Know that $\det(A) = 0$ if and only if $A$ is not invertible.
  - Know properties of determinant: product, transpose, etc.

- **Diagonalization**
  - This is a very, very important topic! Expect some problems on the final on this.
  - Know what eigenvalues/eigenvectors are. Eigenvectors corresponding to distinct eigenvalues are linearly independent!
  - A matrix $A$ is a diagonalizable if it can be written as $A = PDP^{-1}$, where $P$ is an invertible matrix and $D$ is a diagonal matrix.
  - Not all square matrices are diagonalizable!!! To determine whether or not an $n \times n$ matrix $A$ is diagonalizable, find all eigenvalues, and calculate the dimension of the eigenspaces. If the dimensions add up to $n$, the matrix is diagonalizable. If they do not add up to $n$, the matrix is not diagonalizable.
If the matrix $A$ is diagonalizable, the matrix $P$ contains the vectors in the bases of eigenspaces as columns, and $D$ is a diagonal matrix containing the corresponding eigenvalues as columns.

Diagonalization is useful when computing powers of matrices. If a matrix $A$ is diagonalizable, then $A = PDP^{-1}$, and $A^k = PD^kP^{-1}$.

**Orthogonal Projections, Orthogonal and Orthonormal Vectors, Gram-Schmidt Process**

- Know what orthogonal and orthonormal vectors are. Orthogonal sets consisting of non-zero vectors are linearly independent!
- Know what the orthogonal complement is.
- Know the Gram-Schmidt process for converting any basis to an orthogonal basis.
- Orthogonal Projections
  * Know how to write $y = \hat{y} + z$, where $\hat{y} \in W$ and $z \in W^\perp$
  * $\hat{y}$ is called an orthogonal projection, and it is the closest point from $y$ to $W$.
  * The distance from $y$ to $W$ is $||y - \hat{y}||$.

**Matrix Factorizations**

- LU Factorization
  * Know how to compute $LU$ factorization.
  * Know how to use $LU$ factorization to compute the solution to $Ax = b$.
- QR Factorization
  * Know how to compute $QR$ factorization.

**Least Squares Problems**

- The best we can do for an inconsistent system $Ax = b$ is to find $x$ that makes $Ax$ as close as possible to $b$: such an $x$ is called a least squares solution.
- The set of least-squares solutions of $Ax = b$ coincides with the set of solutions of $A^T Ax = A^T b$.
- know how to calculate least-squares error.