Math 220 A (Friday, December 6, 2002)

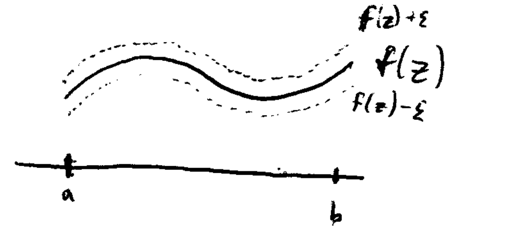
Def Let $\{f_n(z)\}$ and $f(z)$ be functions defined on a region A.

Then $\{f_n(z)\}$ converges uniformly to $f(z)$ if for every $\varepsilon > 0$,

there exists an N s.t. $n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon$ for all $z \in A$.

The difference between this and pointwise convergence is the same N works for every $z \in A$ at the same time.

Picture:

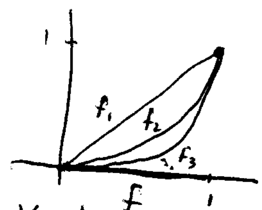


Eventually, all f_n will be within the dotted lines.

To illustrate uniform convergence, it's easier to give examples of sequences which are not uniformly convergent.

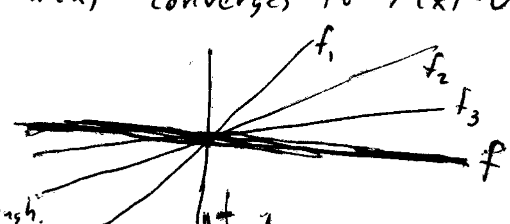
Example 1 $f_n(x) = x^n$ on $[0,1]$. $f_n(x)$ converges pointwise to $f(x) = \begin{cases} 0 & \text{if } x \neq \\ 1 & \text{if } x = 1 \end{cases}$

But $f_n(x)$ does not converge uniformly to $f(x)$.



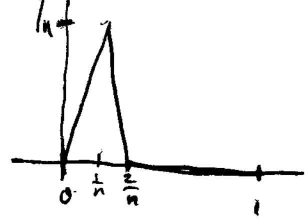
Note $f(x)$ is not continuous. Can a continuous function be the non uniform limit of continuous functions? Yes!

Example 2 $f_n(x) = \frac{1}{n} \cdot x$ on $\mathbb{R}$. $f_n(x)$ converges to $f(x) = 0$ pointwise, but the limit is not uniform.



The limit would be uniform on $[0,1]$, though.

Example 3 $f_n(x) = \begin{cases} n^2 x & \text{for } 0 \leq x \leq \frac{1}{n} \\ 2n - n^2 x & \text{for } \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & \text{for } \frac{2}{n} < x \leq 1 \end{cases}$



$f_n(x) \to 0$

$\int_0^1 f_n(x)\,dx = 1$ for all n but $\int_0^1 f(x)\,dx = 0$.

Thm Let $\{f_n(z)\}$ be a sequence of analytic functions in a region A which contains the disk $D: |z-z_0|\le R$. If $\{f_n(z)\}$ converges uniformly on the boundary of D, then $\{f_n(z)\}$ converges on all of D to a function $f(z)$ which is analytic on the interior of D, and $\{f_n^{(k)}(z)\}$ converges to $f^{(k)}(z)$ in the interior of D. This convergence is uniform on $|z-z_0|\le R-\varepsilon$.

In particular, if $f_n(z)\to f(z)$ uniformly on A, then f is analytic and $f_n^{(k)}(z)\to f^{(k)}(z)$ on A, the convergence being uniform on all compact subsets of A.

Proof Since the $f_n(z)$ converge uniformly to $f(z)$ on the boundary of D, f must be continuous on the boundary of D, in the sense that $f(z_0+Re^{it})$ is continuous with respect to the real variable t.

For any other z on the interior of D, set $f(z)=\dfrac{1}{2\pi i}\displaystyle\int_{|\xi-z_0|=R}\frac{f(\xi)\,d\xi}{\xi-z}$, which is analytic for $|z-z_0|<R$.

Then $f^{(k)}(z)-f_n^{(k)}(z)=\dfrac{k!}{2\pi i}\displaystyle\int_{|\xi-z_0|}\frac{[f(\xi)-f_n(\xi)]}{(\xi-z)^{k+1}}\,d\xi$, which tends to 0 for fixed z as $n\to\infty$, and uniformly for $|z-z_0|\le R-\varepsilon$, as desired.

As for compact sets, take as an open covering open discs such that the closed disc of twice radius is still inside A. Taking a finite subcovering allows us to take the maximum of all the N's for a particular $\varepsilon>0$. ∎

Example Compactness is essential. $f_N(z)=\sum_{n=1}^{N}\frac{z^n}{n^2}$ converges uniformly on $|z|\le 1$, but the derivatives $f_N'(z)=\sum_{n=0}^{N-1}\frac{z^n}{n+1}$ do not converge uniformly on even $|z|<1$ since $f'(z)\to\infty$ as $z\to 1^-$ through real values.

Now apply this theorem to power series.

If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges at $z = z_1$, then the sum is absolutely convergent for all $|z-z_0| < |z_1 - z_0|$ and uniformly convergent on $|z-z_0| \le |z_1 - z_0| - \varepsilon$.

So there exists an R, $0 \le R \le \infty$, called the radius of convergence such that the series converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$.

Thm If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has radius of convergence $R > 0$, then the series converges uniformly on $|z-z_0| \le R - \varepsilon$, and $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)$ is analytic on $|z-z_0| < R$. Furthermore, we can "pull in the derivative":

$$f^{(k)}(z) = \sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1) \cdot \ldots (n-k+1) \, (z-z_0)^{n-k} = \sum_{n=0}^{\infty} a_{n+k} (n+k)(n+k-1)\ldots(n+1) \, z^n$$

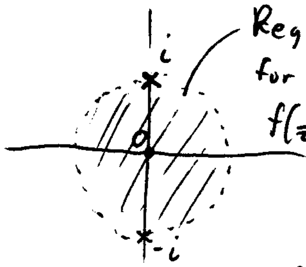
which also has radius of convergence R.

In particular, $f^{(k)}(z_0) = k! \, a_k$, and hence $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is the Taylor series expansion for $f(z)$. Thus, Taylor series (and also Laurent series) are unique.

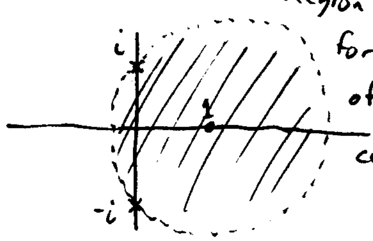
Note: There are different Laurent series for the same function, but convergent in different annuli. The Laurent series is only unique for a given annulus.

Proof Basically apply previous theorem. The only thing is to check that the derivatives have the same radius of convergence. But if the derivative had a larger radius of convergence, it would have a primitive (antiderivative) on this larger region, which differed from our original function by a constant, contradiction.

The radius of convergence is actually determined by the poles of function. For example, $f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ as a real valued function is defined on all of $\mathbb{R}$, but its power series only has Radius of convergence equal to 1. The radius actually comes from the complex analog $f(z) = \frac{1}{1+z^2}$, which has poles at $\pm i$.



Region of convergence for power series of $f(z) = \frac{1}{1+z^2}$ centered at $z_0 = 0$.



Region of convergence for power series of $f(z) = \frac{1}{1+z^2}$ centered at $z_0 = 1$.

In general, the Taylor series centered at z_0 will have radius as "large" as possible, out to the closest pole. Hence the Taylor series for $f(z) = \frac{1}{1+z^2}$ centered at $z_0 = 1$ should have radius of convergence $|1-i| = \sqrt{2} = |1-(-i)|$.

Thm For absolute convergent series, any rearrangement will converge to the same limit.

Cor We can multiply and divide power series where it makes sense.

For example $\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} c_n z^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

By inversion, we can find a power series (or Laurent series) for $\frac{1}{f(z)}$.

Example: Power series for $\tan z = \frac{\sin z}{\cos z}$.

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n!)} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots \quad \text{Let } \frac{1}{\cos z} = \sum_{n=0}^{\infty} a_n z^n.$$

Then $\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right) = 1 + 0z + 0z^2 + \dots$

$\Rightarrow \underline{a_0 = 1}, \ \underline{a_1 = 0} \qquad -\frac{1}{2} + a_2 = 0 \Rightarrow \underline{a_2 = \frac{1}{2}}. \qquad \underline{a_3 = 0}$

$a_4 - \frac{1}{2} a_2 + \frac{1}{24} a_0 = 0 \quad \Rightarrow \underline{a_4 = \frac{1}{4} - \frac{1}{24} = \frac{5}{24}}$

Hence, $\frac{1}{\cos z} = 1 + \frac{z^2}{2} + \frac{5}{24} z^4 + \dots$

and
$$\frac{\sin z}{\cos z} = \left(z - \frac{z^3}{3!} + \frac{z^5}{120} - \dots\right)\left(1 + \frac{z^2}{2} + \frac{5}{24} z^4 + \dots\right)$$
$$= z + \frac{1}{3} z^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right) z^5 + \dots$$
$$= z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \dots$$

In particular, we can use this to evaluate integrals involving $\tan z$.

For example, $\operatorname*{Res}_{z=0} \frac{\tan z}{z^4} = \frac{1}{3}$, so $\underline{\int_{|z|=1} \frac{\tan z}{z^4}\, dz = \frac{2\pi i}{3}}$.

Here is a clarification of a claim made in class.

The claim was that if $\{f_n(z)\}$ is a sequence of analytic functions which converge uniformly on the boundary of the disc $|z-z_0| \le R$, then $f_n^{(k)}(z)$ converges to $f^{(k)}(z)$ uniformly on $|z-z_0| \le R-\delta$, for any $0<\delta<R$. First, we formulate an equivalent condition to uniform convergence.

<u>Lemma</u> $f_n(z) \to f(z)$ uniformly on A iff $\forall \varepsilon>0, \exists N$ s.t. $n \ge N \Rightarrow \sup_{z\in A} |f_n(z)-f(z)| < \varepsilon$

<u>Proof</u> Obvious from definition.

<u>Proof of Claim</u> Fix δ, $0<\delta<R$, and consider any z on the closed disc $|z-z_0| \le R-\delta$. If ξ is a point on the boundary of $|z-z_0| \le R$ (ie., $|\xi - z_0| = R$), then $|\xi - z| \ge \big|\,|\xi-z_0| - |z_0-z|\,\big| \ge |R-(R-\delta)| = \delta$.

Now fix a $k \in \mathbb{Z}^+ \cup \{0\}$. Then $f^{(k)}(z) - f_n^{(k)}(z) = \dfrac{k!}{2\pi i}\displaystyle\int_{|\xi-z_0|=R} \frac{[f(\xi)-f_n(\xi)]}{(\xi-z)^{k+1}}\,d\xi$.

Fix $\varepsilon>0$.

By uniform convergence of $f_n(z)$ to $f(z)$ on $|\xi - z_0| = R$ (boundary of larger disc), choose N large enough such that for $n \ge N$ $\sup_{|\xi-z_0|=R} |f_n(\xi) - f(\xi)| < \dfrac{\delta^{k+1}}{R\cdot k!}\cdot\varepsilon$

Then
$$|f^{(k)}(z) - f_n^{(k)}(z)| \le \frac{k!}{2\pi}\,\frac{\sup_{|\xi-z_0|=R}|f(\xi)-f_n(\xi)|}{\delta^{k+1}}\cdot 2\pi R = \frac{k!\,R}{\delta^{k+1}}\cdot \sup_{|\xi-z_0|=R}|f(\xi)-f_n(\xi)|$$
$$< \frac{k!\,R}{\delta^{k+1}}\cdot\frac{\delta^{k+1}}{R\cdot k!}\,\varepsilon = \varepsilon.$$

Since our choice of z in the disc $|z-z_0| \le R-\delta$, the convergence is uniform there. ∎

If we attempted the same proof for $|z-z_0|<R$, we would not be able to bound the $(\xi - z)^{k+1}$ term effectively, and the whole proof would fall apart. So convergence on $|z-z_0|<R$ need not be uniform.

Complex Analysis Trivia

① What is an example of a complex-valued function on $\mathbb{C}$ which is continuous everywhere but differentiable nowhere? Ans: $f(z) = \bar{z}$

② What's an example of an $f(z)$ which has a derivative at exactly one point? Ans: $f(z) = |z|^2$

③ Example of: (a) Isolated essential singularity? Ans: $f(z) = e^{\frac{1}{z}}$ or $f(z) = \sin\left(\frac{1}{z}\right)$

(b) Infinite number of isolated singularities? Ans: $f(z) = \tan z$, $\sec z$, etc.

(c) A nonisolated singularity? Ans: $f(z) = \dfrac{1}{\sin\left(\frac{1}{z}\right)}$ at $z=0$.