Problem 1
Consider the following first order differential equation:
\[ y' + y = y^2, \quad y(0) = \frac{1}{2} \]
(1) Is this equation linear or not?
(2) Use the substitution \( v = \frac{1}{y} \) to reduce this equation into linear equation and solve \( y \).

Problem 2
Consider the following first order differential equation:
\[ (3xy + y^2) + (x^2 + xy)y' = 0 \]
Is this equation exact or not? If yes, explain why and solve it. If not, find an integrating factor and solve it. Leave the solution in the implicit form.

Problem 3
Solve the following initial value problem
\[ y' = \sqrt{y}, \quad y(0) = 0 \]
Is the solution unique? If not, why it fails to have unique solution?

Problem 4
Consider the following second order homogeneous equation:
\[ x^2 y'' + 3xy' + y = 0, \quad x > 0 \]
(1) Justify \( y_1(x) = x^{-1} \) is a solution.
(2) Find another fundamental solution using the method of reduction of order. Be sure to explain why your solution is another fundamental solution.

Problem 5
Consider the following second order nonhomogeneous equation:
\[ y'' - 3y' - 4y = -5e^{-x} + 8x^2 - 13 \]
Find the general solution of this differential equation.

Problem 6
Solve the following initial value problem:
\[
x' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}
\]

Problem 7
Consider the following system of equations:

\[
x' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}
\]

(1) Compute the fundamental matrix \( \Psi(t) \) and compute its inverse.
(2) Find the general solution of the system

Problem 8
Consider the power series solution of the following equation about the given point \( x_0 \):

\[
y'' - y = 0, \quad x_0 = 0
\]

(1) Write down the recursion formula.
(2) Identify the two fundamental solutions \( y_1 \) and \( y_2 \) as power series.

Problem 9
Use the Laplace transform to solve the following initial value problem.

\[
y'' - 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = 4
\]

Note: you must use Laplace transform, otherwise you will get 0.

Problem 10
In this final exam, one identical problem will be picked from examples of classnotes and one identical problem will be picked from homework (HW1-HW5).

It is advised that you go over concepts, summaries, conclusions and examples of classnotes and make sure you are able to solve homework independently.