CHAPTER 2

Ramsey theory

2.1. Introduction

In this chapter, we will survey graph (and hypergraph) problems of Paul Erdős (often with his collaborators) arising out of his work in Ramsey theory. The guiding philosophy in this subject deals with the inevitable occurrence of specific structures in some part of a large arbitrary structure which has been partitioned into finitely many parts. Well-known examples of this are the Pigeonhole Principle, van der Waerden's theorem on arithmetic progressions and Ramsey's theorem itself. We will say more about these in subsequent sections.

2.2. Origins

Paul's first results in this area occurred in his joint paper [1], written with George Szekeres and published in 1935. Simply titled, "A combinatorial problem in geometry", it laid the groundwork for an amazing variety of subsequent work during the next 60 years. This question arose out of a question posed by Esther Klein, a talented young mathematician in Budapest, who asked:

Is it true that for all n, there is a least integer $g(n)$ so that any set of $g(n)$ points in the plane in general position must always contain the vertices of a convex n-gon?

She had previously observed that $g(4) = 5$. The reader is encouraged to read Szekeres' touching accounts [2] [3] of how this joint work arose, and the effects it had on his life and career (in particular, he married Esther Klein the following year, in 1936, and they remain still happily married, living and working in Australia now. This is the reason Paul often referred this affirmative solution to Esther Klein's question as the "Happy End" theorem.)

In proving that $g(n)$ exists, Szekeres actually rediscovered Ramsey's theorem, which had only appeared (unknown to him then) some five years earlier. Erdős and

[1] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463–470.

[2] *Paul Erdős, The Art of Counting*, ed. Joel Spencer, The MIT Press, Cambridge, Massachusetts, 1973.

[3] P. Erdős, Some of my favorite problems and results, *The Mathematics of Paul Erdős* (R. L. Graham and J. Nešetřil, eds.), 47–67, Springer-Verlag, Berlin, 1996.

Szekeres established the following bounds on $g(n)$:

$$2^{n-2} + 1 \le g(n) \le \binom{2n-4}{n-2} + 1. \tag{2.1}$$

They further conjectured that the lower bound is actually the correct answer.

The proof for (2.1) is based on several interrelated fundamental facts which illustrate the spirit of Ramsey theory:

(i) *For any sequence of n^2+1 distinct numbers, say, $x_1, x_2, \ldots, x_{n^2+1}$, there is always either an increasing subsequence (i.e., $x_{i_1} \le x_{i_2} \le \ldots \le x_{i_{n+1}}$ with $i_1 < i_2 < \ldots < i_{n+1}$) of $n+1$ numbers, or a decreasing subsequence (i.e., $x_{j_1} \ge x_{j_2} \ge \ldots \ge x_{j_{n+1}}$ with $j_1 < j_2 < \ldots < j_{n+1}$) of length $n+1$.*

(ii) *For given positive integers m and n, any set of $\binom{n+m-2}{n-1} + 1$ points in general position in the plane must contain either n points $x_1, \ldots, x_n$ with consecutive line segments $x_i x_{i+1}$ of increasing slopes, or m points with consecutive line segments of decreasing slopes.*

Both (i) and (ii) have short elegant proofs which are perhaps the *Book Proofs*. In Erdős' language, those proofs belong to the Book (containing the best possible proofs of each theorem in mathematics), which we mortals can only occasionally get a glimpse of.

Proof of (i): We associate to each number x_j, a pair of integers (a_j, b_j) where a_j denotes the length of the longest increasing subsequence ending at x_j, and b_j denotes the length of the longest decreasing subsequence ending at x_j. It is easy to see that $(a_i, b_i) \ne (a_j, b_j)$ for $i \ne j$. Since there are n^2+1 numbers x_j, not all the (a_j, b_j) can satisfy $a_j, b_j \le n$. Thus, there is a monotone subsequence of length at least $n+1$. □

Proof of (ii): Let $f(n,m)$ denote the maximum number of points such that there is no n-cup (i.e., n points with consecutive line segments having increasing slopes), and there is no m-cap (i.e., m points with consecutive line segments having decreasing slopes). It suffices to show

$$f(n,m) \le f(n,m-1) + f(n-1,m).$$

Suppose S is a set of $f(n,m)$ points containing no n-cup and no m-cap. We consider the set T of points x which are the right endpoints of some $(n-1)$-cup. Clearly, x cannot be the left endpoint of an $(m-1)$-cap. Therefore, we have

$$|T| \le f(n,m-1).$$

Also,

$$|S \setminus T| \le f(n-1,m).$$

This proves (ii). □

Now the upper bound for $g(n)$ follows immediately from (ii) since an n-cup or n-cap forms a convex n-gon.

The lower bound for $g(n)$ in (2.1) is established by appropriately combining sets of sizes $f(\lfloor n/2 \rfloor - 2i, \lfloor n/2 \rfloor + 2i)$ for integers i in the interval $(-\lfloor n/2 \rfloor, \lceil n/2 \rceil)$.

Conjecture

$$g(n) = 2^{n-2} + 1 \tag{2.2}$$

for all n.

This is known to hold for $n = 3, 4$ and 5. The upper bound had remained unchanged for some sixty years until very recently, when the following improvement was proved [4].

$$g(n) \leq \binom{2n-4}{n-2}, \text{ for } n \geq 4. \tag{2.3}$$

This improvement, although microscopic, has triggered a flurry of activity, including a new bound $g(n) \leq \binom{2n-4}{n-2} - 2n + 7$ by Kleitman and Pachter [5], which was further improved by Tóth and Valtr [6] to $g(n) \leq \binom{2n-5}{n-3} + 2$. Clearly, there is still plenty of room for further improvement.

Let us call a convex polygon P formed from the points of a set X *empty* if the interior of P contains no point of X. Erdős suggested the following variation.

For each n, let $g^*(n)$ denote the least integer such that any set of $g^*(n)$ points in the plane in general position must always contain the vertices of an empty convex n-gon. Is it true that $g^*(n)$ always exists, and if it does, determine or estimate $g^*(n)$.

While $g^*(3) = 3, g^*(4) = 5$ and $g^*(5) = 10$ (see Harborth[7]), it was shown unexpectedly by Horton [8] that $g^*(7) = \infty$. That is, there is an infinite set in the plane in general position containing no empty 7-gon. At the time of this writing, the situation for $g^*(6)$ is still completely open.

A weaker restriction in this vein has been considered by Bialostocki, Dierker and Voxman [9]. They prove that there is a function $E(n, q)$ such that if X is a subset of the plane in general position with $|X| \geq E(n, q)$, then X always contains the vertices of a convex n-gon with tq points of X in its interior for some integer t, where $n \geq q + 2$. Caro[10] shows that one can always take $E(n, q) \leq 2^{c(q)n}$ where $c(q)$ depends only on q.

---

[4]F. R. K. Chung and R. L. Graham, Forced convex n-gons in the plane, *Discrete and Computational Geometry*, to appear.

[5]D. J. Kleitman and L. Pachter, Finding convex sets among points on the plane, *Discrete and Computational Geometry*, to appear.

[6]G. Tóth and P. Valtr, Note on the Erdős-Szekeres problem, preprint.

[7]H. Harborth, Konvexe Fünfecke in ebenen Punktmengen, *Elem. Math.* **33** (1978), 116-118.

[8]J. D. Horton, Sets with no empty convex 7-gons, *Canad. Math. Bull.* **26**, (1983), 482-484.

[9]A. Bialostocki, P. Dierker and W. Voxman, Some notes on the Erdős-Szekeres theorem, *Discrete Math.* **91** (1991), 117-127.

[10]Y. Caro, On the generalized Erdős-Szekeres Conjecture– a new upper bound, *Discrete Math.* **160** (1996), 229-233.

2.3. Classical Ramsey Theory

Here we state the simple version of Ramsey's theorem for coloring graphs in two colors. The original statement is much more general. The versions for hypergraphs, infinite graphs and/or with more colors will be discussed in later sections.

For two graphs G and H, let $r(G, H)$ denote the smallest integer m satisfying the property that if the edges of the complete graph K_m are colored in red or blue, then there is either a subgraph isomorphic to G with all red edges or a subgraph isomorphic to H with all blue edges.

The classical Ramsey numbers are those for the complete graphs and are denoted by $r(s,t) = r(K_s, K_t)$. In the special case that $n_1 = n_2 = n$, we simply write $r(n)$ for $r(n,n)$, and we call this the *Ramsey number* for K_n.

2.3.1. On Ramsey numbers for K_n.

The problem of accurately estimating $r(n)$ is a notoriously difficult problem in combinatorics. The only known values [11] are $r(3) = 6$ and $r(4) = 18$. For $r(5)$, the best bounds [12,13]. are $43 \leq r(5) \leq 49$. For the general $r(n)$, the earliest bounds were:

$$\frac{1}{e\sqrt{2}} n 2^{n/2} < r(n) \leq \binom{2n-2}{n-1}. \tag{2.4}$$

The upper bound follows from the fact that the Ramsey number $r(k,l)$ satisfies:

$$r(k,l) \leq r(k-1,l) + r(k,l-1) \tag{2.5}$$

with strict inequality if both $r(k-1,l)$ and $r(k,l-1)$ are even. To see this, if $n = r(k-1,l) + r(k,l-1)$, for any vertex v, there are either at least $r(k-1,l)$ red edges or at least $r(k,l-1)$ blue edges leaving v. Therefore there is either a red copy of K_k or a blue copy of K_l. The strict inequality condition is a consequence of the fact that a graph on an odd number of vertices can not have all odd degrees.

The lower bound is established by a counting argument given by Erdős in [14], which can be described as follows:

There are $2^{\binom{m}{2}}$ ways to color the edges of K_m in 2 colors. The number of colorings that contain a monochromatic K_n is at most

$$\binom{m}{n} 2^{\binom{m}{2} - \binom{n}{2} + 1}.$$

Therefore, there exists a coloring containing no monochromatic K_k if

$$2^{\binom{m}{2}} > \binom{m}{n} 2^{\binom{m}{2} - \binom{n}{2} + 1}.$$

[11] R. E. Greenwood and A. M. Gleason, Combinatorial relations and chromatic graphs, *Canad. J. Math.* **7** (1955), 1-7.

[12] G. Exoo, A lower bound for $R(5,5)$, *J. Graph Theory 13* (1989), 97-98.

[13] B. D. McKay and S. P. Radziszowski, Subgraph counting identities and Ramsey numbers, *J. Comb. Theory* (B), **69** (1997), 193-209.

[14] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

This is true when

$$m \geq \frac{1}{e\sqrt{2}} n 2^{n/2}.$$

So, the lower bound in (2.4) is proved.

Very little progress has occurred in the intervening fifty years in improving these bounds. The best current bounds are

$$\frac{\sqrt{2}}{e} n 2^{n/2} < r(n) < n^{-1/2+c/\sqrt{\log n}} \binom{2n-2}{n-1}. \tag{2.6}$$

The upper bound is due to Thomason [15] and the lower bound is due to Spencer [16] by using the Lovász local lemma, which we will describe here.

The Lovász local lemma

Let $A_1, \ldots, A_q$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all but at most d of the other events A_j, and that $Pr(A_i) \leq p$ for all $1 \leq i \leq q$. If

$$ep(d+1) \leq 1,$$

then $Pr(\bigwedge_{i=1}^{q} \bar{A}_i) > 0$.

For each set S of n vertices in a graph with m vertices, let A_S denote the event that the complete graph on S is monochromatic. Therefore, $Pr(A_S) = 2^{1-\binom{n}{2}} = p$. Since each event A_S is mutually independent of all the events A_T satisfying $|S \cap T| \leq 1$, we have $d = \binom{n}{2}\binom{m}{n-2}$. Using the Lovász local lemma, if

$$e(\binom{n}{2}\binom{m}{n-2} + 1) 2^{1-\binom{n}{2}} < 1,$$

we have $r(n,n) > m$. A straightforward simplification gives

$$r(n) > \frac{\sqrt{2}}{e} n 2^{n/2}.$$

In particular, we see that $r(n)^{1/n}$ lies between $\sqrt{2}$ and 4.

Conjecture \$100 (1947)
The limit

$$\lim_{n\to\infty} r(n)^{1/n} \tag{2.7}$$

exists.

Problem \$250 (1947)
Determine the value of

$$c := \lim_{n\to\infty} r(n)^{1/n} \tag{2.8}$$

if it exists.

---

[15]A. Thomason, An upper bound for some Ramsey numbers, *J. Graph Theory* **12** (1988), 509–517.

[16]J. Spencer, Ramsey's theorem—a new lower bound, *J. Comb. Theory Ser. A* **18** (1975), 108–115.

2.3.2. On constructing Ramsey graphs. The known lower bounds for $r(n)$ are proved non-constructively, i.e., by using the probabilistic method. It would be very desirable to have an explicit construction giving a similar bound for $r(n)$. This motivates the next problem.

> *A problem on constructive Ramsey bounds* ($100)
> Give a constructive proof that
> (2.9) $$r(k) > (1+c)^k$$
> for some $c > 0$.

In other words, construct a graph on n vertices which does not contain any clique of size $c' \log n$ and does not contain any independent set of size $c' \log n$.

Attempts have been made over the years to construct Ramsey graphs (i.e., with small cliques and independent sets) without much success. Abbott [17] gave a recursive construction with cliques and independence sets of size $cn^{\log 2/\log 5}$. Nagy [18] gave a construction reducing the size to $cn^{1/3}$. A breakthrough finally occurred several years ago with the result of Frankl [19] who gave the first Ramsey construction with cliques and independent sets of size smaller than n^ϵ for any $\epsilon > 0$. This was further improved to $e^{c(\log n)^{3/4}(\log\log n)^{1/4}}$ in [20]. Here we will outline a construction of Frankl and Wilson [21] for Ramsey graphs with cliques and independent sets of size at most $e^{c(\log n \log\log n)^{1/2}}$. In other words,

(2.10) $$r(k) > k^{c\log k/\log\log k}.$$

This construction is based upon a beautiful theorem on set intersections due to Frankl and Wilson[21]:

Theorem

Let p denote a prime power and suppose $\mu_0, \mu_1, \ldots, \mu_s$ are distinct non-zero residues modulo p. We consider a family $\mathcal{F}$ consisting of k-sets of an n-set with the property that for all $S, T \in \mathcal{F}$, we have $|S \cap T| \equiv \mu_i$ (mod p) for some i, $0 \le i \le s$. Then

$$|\mathcal{F}| \le \binom{n}{s}.$$

Now, we consider the graph G which has vertex set $V = \{F \subseteq \{1, \cdots, m\} : |F| = q^2 - 1\}$ and edge set $E = \{(F, F') : |F \cap F'| \not\equiv -1 (\text{mod } q)\}$. The above theorem

[17] H. L. Abbott, Lower bounds for some Ramsey numbers, *Discrete Math.* **2** (1972), 289-293.

[18] Zs. Nagy, A constructive estimation of the Ramsey numbers, *Mat. Lapok* **23** (1975), 301-302.

[19] P. Frankl, A constructive lower bound for some Ramsey numbers, *Ars Combinatoria* **3** (1977), 297-302.

[20] F. R. K. Chung, A note on constructive methods for Ramsey numbers, *J. Graph Th.* **5** (1981), 109-113.

[21] P. Frankl and R. M. Wilson, Intersection theorems with geometric consequences, *Combinatorica* **1** (1981), 357-368.

implies that G contains no clique or independent set of size $\binom{m}{q-1}$. By choosing $m = q^3$, we obtain a graph on $n = \binom{m}{q^2-1}$ vertices containing no clique or independent set of size $e^{c(\log n \log\log n)^{1/2}}$.

In the past ten years, there has been a great deal of development in explicit constructions of so-called *expander graphs*. (which are graphs with certain isoperimetric properties). In particular, Lubotzky, Phillips and Sarnak [22] and Margulis[23] [24] [25] have successfully obtained explicit constructions for expander graphs. However, we are still quite far away from constructing Ramsey graphs on n vertices which contain no clique of size $c \log n$ and no independent set of size $c \log n$.

2.3.3. Off-diagonal Ramsey numbers. For off-diagonal Ramsey numbers, the additional known values are $r(3,4) = 9$, $r(3,5) = 14$, $r(3,6) = 18$, $r(3,7) = 23$, $r(3,8) = 28$, $r(3,9) = 36$ and $r(4,5) = 25$ while $35 \le r(4,6) \le 41$ (see the dynamic survey of Radzisowski on small Ramsey numbers in the *Electronic Journal of Combinatorics,* at www.combinatorics.org for more bounds and references).

For $k = 3$, Kim [26] recently proved a new lower bound which matches the previous upper bound for $r(3,n)$ (up to a constant factor), so it is now known that

$$\frac{cn^2}{\log n} < r(3,n) < (1+o(1))\frac{n^2}{\log n}. \tag{2.11}$$

Ajtai, Komlós and Szemerédi [27] earlier gave the upper bound of $c' \frac{n^2}{\log n}$ and Shearer [28] [29] replaced c' by $1 + o(1)$. It would be of interest to have an asymptotic formula for $r(3,n)$.

The best known constructive lower bound for $r(3,n)$ is due to Alon [30]

$$r(3,n) \ge cn^{3/2}.$$

[22] A. Lubotzky, R. Phillips and P. Sarnak, Ramanujan graphs, *Combinatorica* **8** (1988), 261–277.

[23] G. A. Margulis, Arithmetic groups and graphs without short cycles, *6th Internat. Symp. on Information Theory, Tashkent* (1984) *Abstracts* **1**, 123-125 (in Russian).

[24] G. A. Margulis, Some new constructions of low-density parity check codes, *3rd Internat. Seminar on Information Theory, convolution codes and multi-user communication, Sochi* (1987), 275-279 (in Russian).

[25] G. A. Margulis, Explicit group theoretic constructions of combinatorial schemes and their applications for the construction of expanders and concentrators, *Problemy Peredaci Informacii* (1988) (in Russian).

[26] J. H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Structures and Algorithms* **7** (1995), 173–207.

[27] M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey numbers, *J. Comb. Theory Ser. A* **29** (1980), 354–360.

[28] J. Shearer, A note on the independence number of triangle-free graphs, *Discrete Math.* **46** (1983), 83-87.

[29] J. Shearer, A note on the independence number of triangle-free graphs II, *J. Comb. Theory* (B) **53** (1991), 300-307.

[30] N. Alon, Explicit Ramsey graphs and orthonormal labellings, *Elec. J. Comb.* **1** (1994), R12 (8pp).

improving previous bounds of Erdős[31] and others [32].

For $r(4,n)$, the best lower bound known is $c(n \log n)^{5/2}$ due to Spencer, [33] again by using the Lovász local lemma. The best upper bound known is $c'n^3/\log^2 n$, proved by Ajtai, Komlós and Szemerédi[27]. So there is a nontrivial gap still remaining, as repeatedly pointed out in many problems papers [34] of Erdős.

Problem [19] ($250)
Prove or disprove that

$$r(4,n) > \frac{n^3}{\log^c n} \tag{2.12}$$

for some c, provided n is sufficiently large.

For general k, the best asymptotic bounds for $r(k,n)$, for n large, are as follows:

$$c\left(\frac{n}{\log n}\right)^{(k+1)/2} < r(k,n) < (1+o(1))\frac{n^{k-1}}{\log^{k-2} n}. \tag{2.13}$$

The upper bound is a recent result of Li and Rousseau [35] who extend Shearer's method to improve the constant factor for the bounds in [27]. The lower bound is given in [33].

Conjecture (1947)
For fixed k,

$$r(k,n) > \frac{n^{k-1}}{\log^c k} \tag{2.14}$$

for a suitable constant $c > 0$ and n large.

Very few results are known about the gaps between 'consecutive' Ramsey numbers. Here are several problems appearing in the 1981 problem paper[36].

Problem (Burr and Erdős[36])
Prove that

$$r(n+1,n) > (1+c)r(n,n) \tag{2.15}$$

for some fixed $c > 0$.

---

[31] P. Erdős, On the construction of certain graphs, *J. Comb. Theory* **17** (1966), 149-153

[32] F. R. K. Chung, R. Cleve and P. Dagum, A note on constructive lower bounds for the Ramsey numbers $R(3,t)$

[33] J. Spencer, Asymptotic lower bounds for Ramsey functions, *Discrete Math.* **20** (1977/78), 69–76.

[34] P. Erdős, Problems and results on graphs and hypergraphs: similarities and differences, *Mathematics of Ramsey theory, Algorithms Combin., 5*, (J. Nešetřil and V. Rödl, eds.), 12–28, Springer, Berlin, 1990.

[35] Y. Li and C. C. Rousseau, Bounds for independence numbers and classical Ramsey numbers, preprint.

[36] P. Erdős, Some new problems and results in graph theory and other branches of combinatorial mathematics, *Combinatorics and graph theory (Calcutta, 1980), Lecture Notes in Math., 885*, 9–17, Springer, Berlin-New York, 1981.

We know from (2.5) that $r(3, n+1) \leq r(3, n) + n$. In [36], Erdős said, "Faudree, Schelp, Rousseau and I needed recently a lemma stating

$$\frac{r(n+1, n) - r(n, n)}{n} \to \infty$$

as $n \to \infty$. We could prove this without much difficulty, but we could not prove that $r(n+1, n) - r(n, n)$ increases faster than any polynomial in n. We of course expect

$$\lim_{n\to\infty} \frac{r(n+1, n)}{r(n, n)} = C^{1/2}$$

where

$$C = \lim_{n\to\infty} r(n, n)^{1/n}.$$

V. T. Sós and I recently needed the following results ..."

Conjecture (proposed by Erdős and Sós[36])

(2.16) $$r(3, n+1) - r(3, n) \to \infty \ , \ \text{for } n \to \infty.$$

Conjecture (proposed by Erdős and Sós[36])
Prove or disprove that

(2.17) $$r(3, n+1) - r(3, n) = o(n).$$

This conjecture remains unresolved even with the knowledge of Kim's recent results on $r(3, n)$ (see (2.11)).

2.4. Graph Ramsey theory

Because of the early realization of the difficulty in obtaining sharp results for the classical Ramsey numbers, focus turned to the general study of the numbers $r(G, H)$, for *arbitrary* graphs (as opposed to complete graphs). When $G = H$, we write $r(G) = r(G, G)$. There was an initial hope one could gain a better understanding of $r(k, l)$ by working up to complete graphs from various subgraphs of complete graphs. While this goal has not been met, a beautiful theory has emerged which has taken on a life of its own. In gathering references for a book on this topic, its authors Burr, Faudree, Rousseau and Schelp have so far collected over a thousand references. Much of the impetus of this work was due to Erdős. In this section, we describe a number of his favorite problems in this topic.

2.4.1. On Ramsey numbers for bounded degree graphs. Among the most interesting problems on graph Ramsey theory are the linear bounds for graphs with certain upper bound constraints on the degrees of the vertices. In 1975, Erdős [37] raised the problem of proving $r(G) \leq c(\Delta)\, n$ for a graph G on n vertices with bounded maximum degree Δ.

[37] S. A. Burr and P. Erdős, On the magnitude of generalized Ramsey numbers for graphs, *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I; Colloq. Math. Soc. János Bolyai, Vol. 10*, 215–240, North-Holland, Amsterdam, 1975.

This original problem has been settled in the affirmative by Chvátal, Rödl, Szemerédi and Trotter [38]. Their proof is a beautiful illustration of the power of the regularity lemma of Szemerédi.

Roughly speaking, the regularity lemma says that for *any* graph G, we can partition G into a relatively small number of parts such that the bipartite graph between most pairs of parts behaves like a random graph. To be specific, a bipartite graph with vertex set $A \cup B$ is said to be ϵ-regular if for any $X \subset A$ and $Y \subset B$ with $|X| \geq \epsilon|A|, |Y| \geq \epsilon|B|$, the edge density in the induced subgraph $X \cup Y$ is essentially the same as the edge density in $A \cup B$ (differs by at most an additive term of ϵ).

The bounded number of parts depends only on ϵ and is independent of the size of G. The main part of the proof [38] is accomplished by repeatedly using the ϵ-regular property to find a desired monochromatic subgraph. (For an excellent survey article on the regularity lemma and its many applications, the reader is referred to Komlós and Simonovits [39]).

As is typical when using the regularity lemma, the constant $c(\Delta)$ obtained by Chvátal et al.[38] was rather large (more precisely, it had the form of an exponential tower of 2's of height Δ). More recently, Eaton[40] used a variant of the regularity lemma to show that one can take

$$c(\Delta) < 2^{2^{c\Delta}}$$

for some $c > 0$. Subsequently, Graham, Rödl and Ruciński[41] showed that it is enough to take

$$c(\Delta) < 2^{c\Delta(\log \Delta)^2}$$

for some $c > 0$ and $\Delta > 1$. They also show that there are graphs G with n vertices and maximum degree Δ for which $r(G) \geq c_0^{\Delta}\, n$ for some $c_0 > 1$ and n sufficiently large.

Chen and Schelp [42] extended the result by Chvátal et al.[38] replacing the bounded degree condition by the following weaker requirement. A graph is said to be *c-arrangeable* if the vertices can be ordered, say, $v_1, \ldots, v_n$, such that for each i,

$$|\{j : v_i \sim v_k, \text{ for } k > i, \text{ and } v_k \sim v_j \text{ for } j \leq i\}| \ \leq\ c.$$

check the definition

Chen and Schelp proved that for a fixed c, the Ramsey number for c-arrangeable graphs grows linearly with the size of the graph. They showed that a planar graph

[38] V. Chvátal, V. Rödl, E. Szemerédi and W. T. Trotter, The Ramsey number of a graph with bounded maximum degree, *J. Comb. Theory Ser. B* **34** (1983), 239–243.

[39] J. Komlós and M. Simonovits, Szemerédi's regularity lemma and its applications in graph theory, *Combinatorics, Paul Erdős is Eighty*, Vol. 2, (D. Miklós, V. T. Sós, T. Szőnyi, eds.), Bolyai Soc. Math. Studies, 2 (1996), 97–132.

[40] N. Eaton, Ramsey numbers for sparse graphs, *Discrete Math.*, to appear

[41] R. L. Graham, V. Rödl and A. Ruciński, On graphs with linear Ramsey numbers, preprint.

[42] G. Chen and R. H. Schelp, Graphs with linearly bounded Ramsey numbers, *J. Comb. Theory Ser. B* **57** (1993), 138–149.

is 761-arrangeable, which was later improved to 10-arrangeable by Kierstead and Trotter[43] So, their results imply that planar graphs have linear Ramsey numbers.

Recently, Rödl and Thomas[44], generalizing results in [42], showed that graphs with bounded genus have linear Ramsey numbers. The following three problems are in fact equivalent (subject to different constants).

Conjecture on Ramsey numbers for subgraphs with bounded average degrees
(proposed by Burr and Erdős[37])
For every graph G on n vertices in which every subgraph has average degree at most c,

$$r(G) \leq c'n$$

where the constant c' depends only on c.

Conjecture on Ramsey numbers for bounded arboricity
(proposed by Burr and Erdős[37])
If a graph G on n vertices is the union of c forests, then

$$r(G) \leq c'n$$

where the constant c' depends only on c.

Conjecture on Ramsey numbers for graphs with degree constraints
(proposed by Burr and Erdős[37])
For every graph G on n vertices in which every subgraph has minimum degree at most c,

$$r(G) \leq c'n$$

where the constant c' depends only on c.

2.4.2. On relating graph Ramsey numbers to the classical Ramsey problems. The following several problems run along the lines of attempting to clarify the relationship between graph Ramsey numbers and the classical ones. Although these problems [45] [46] were raised very early on, little progress has been made so far.

[43]H. A. Kierstead and W. T. Trotter, Planar graph colorings with an uncooperative partner, *J. Graph Theory* **18** (1994), 569-584.

[44]V. Rödl and R. Thomas, Arrangeability and clique subdivisions, *The Mathematics of Paul Erdős, II* (R. L. Graham and J. Nešetřil, eds.), 236-239, Springer-Verlag, Berlin, 1996.

[45]P. Erdős and R. L. Graham, On partition theorems for finite graphs, *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I; Colloq. Math. Soc. János Bolyai, Vol. 10*, 515–527, North-Holland, Amsterdam, 1975.

[46]P. Erdős, On some problems in graph theory, combinatorial analysis and combinatorial number theory, *Graph theory and combinatorics (Cambridge, 1983)*, 1–17, Academic Press, London-New York, 1984.

Conjecture (proposed by Erdős and Graham [79])
If G has $\binom{n}{2}$ edges for $n \geq 4$, then

$$r(G) \leq r(n).$$

More generally, if G has $\binom{n}{2} + t$ edges, then

$$r(G) \leq r(H)$$

where H denotes the graph formed by connecting a new vertex to t of the vertices of a K_n, and $t \leq n$.

Problem [51]
Is it true that if a graph G has e edges, then

$$r(G) < 2^{ce^{1/2}}$$

for some absolute constant c?

For a graph G, the chromatic number $\chi(G)$ is the least integer k such that the vertices of G can be colored in k colors so that adjacent vertices have different colors. If $\chi(G) \leq k$, we say that G is k-colorable. The following problems [47] relate Ramsey numbers to chromatic numbers.

A problem on k-chromatic graphs [47]
Let G denote a graph on n vertices with chromatic number k. Is it true that

$$r(G) > (1 - \epsilon)^k r(k)$$

holds for any fixed ϵ, $0 < \epsilon < 1$, provided n is large enough?

Problem [47]

Prove that there is some $\epsilon > 0$ so that for all G with chromatic number k,

$$\frac{r(G)}{r(k)} > \epsilon.$$

This is a modified version of an old conjecture that $r(G) \geq r(k)$ which, however, has a counterexample for the case of $n = 4$. It was given by Faudree and McKay [48] by showing $r(W) = 17$ for the pentagonal wheel W.

2.4.3. On Ramsey numbers involving trees. Many Ramsey numbers have been determined for special families of graphs, including various combinations of paths, trees, stars and cycles. However, the following problem [49] on Ramsey numbers for trees is still open.

[47] P. Erdős, Some of my favourite problems in number theory, combinatorics, and geometry, Combinatorics Week (Portuguese) (São Paulo, 1994), *Resenhas* **2** (1995), 165–186.

[48] R. Faudree and B. McKay, A conjecture of Erdős and the Ramsey number $r(W_6)$, *J. Combinatorial Math. and Combinatorial Computing* **13** (1993), 23–31.

[49] S. A. Burr and P. Erdős, Extremal Ramsey theory for graphs, *Utilitas Math.* **9** (1976), 247-258.

Conjecture (proposed by Burr and Erdős [49])
For any tree T on n vertices,
$$r(T) \leq 2n - 2.$$

Clearly, for a star on n vertices, equality holds. So, the above conjecture can be restated as $r(T) \leq r(S_n)$ where S_n denotes the star on n vertices.

The above problem is closely related to a conjecture by Erdős and Sós which will be discussed later in the chapter on extremal graph problems. This conjecture asserts that every graph with m vertices and more than $(n-2)m/2$ edges contains every tree T on n vertices. If this conjecture were true, it would imply the above conjecture.

Suppose that a tree T has a 2-coloring with k vertices in one color and l vertices in the other. It was proved in [50] that
$$r(T) \geq \max\{2k + l - 1, 2l - 1\}.$$
This leads to the following:

Problem [50]
Is $r(T) = 4k$ for every tree which is a bipartite graph with k vertices in one color and $2k$ vertices in the other?

Chvátal [51] proved that
$$r(T, K_m) = (m-1)(n-1) + 1$$
for any tree on n vertices. This result was generalized to graphs with small chromatic number. For a graph G with chromatic number $\chi(G)$, it was shown [52] that
$$r(T, G) = (\chi(G) - 1)(n-1) + 1$$
for any tree T on n vertices, provided n is sufficiently large.

Conjecture [53]
If $m_1 \leq \ldots \leq m_k$, then
$$r(T, K_{m_1,\ldots,m_k}) \leq (\chi(G) - 1)(r(T, K_{m_1,m_2}) - 1) + m_1$$
where T is any tree on n vertices, and n is large enough.

2.4.4. On Ramsey numbers involving cycles.

Conjecture [54]
For some $\epsilon > 0$,
$$r(C_4, K_n) = o(n^{2-\epsilon}).$$

[50] P. Erdős, R. Faudree, C. C. Rousseau and R. H. Schelp, Ramsey numbers for brooms, Proceedings of the thirteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1982), *Congr. Numer.* **35** (1982), 283–293.

[51] V. Chvátal, Tree-complete graph Ramsey numbers, *J. Graph Theory* **1** (1977), 93.

[52] S. A. Burr, P. Erdős, R. J. Faudree, R. J. Gould, M. S. Jacobson, C. C. Rousseau and R. H. Schelp, Goodness of trees for generalized books, *Graphs Combin.* **3** (1987) no. 1, 1–6.

It is known that

$$c(\frac{n}{\log n})^2 > r(C_4, K_n) > c(\frac{n}{\log n})^{3/2}$$

where the lower bound is proved by probabilistic methods[33], and the upper bound is due to Szemerédi (unpublished[55]).

For k fixed and n large, the probabilistic method gives

$$r(C_k, K_n) > c(n/\log n)^{(k-1)/(k-2)}.$$

For the upper bound, it is known [55] [56] that for even k, we have

$$r(C_k, K_n) \leq c_k(n/\log n)^{1+1/m}$$

where $m = k/2 - 1$.

For C_k, with k large compared to n, the Ramsey number $r(C_k, K_n)$ was obtained by Bondy and Erdős[57] :

$$r(C_k, K_n) = (k-1)(n-1) + 1$$

for $k > n^2 - 2$.

Erdős, Faudree, Rousseau, Schelp[55] proposed the following problems:

Problem:
Is it true that

$$r(C_k, K_n) = (k-1)(n-1) + 1$$

if $k \geq n > 3$?

Problem:
What is the smallest value of k such that $r(C_k, K_n) = (k-1)(n-1) + 1$?

Problem:
For a fixed n, what is the minimum value of $r(C_k, K_n)$ over all k?

Together with Burr [58], they proposed the following problem:

Problem
Determine $r(C_4, K_{1,n})$.

It is known that

$$n + \lceil\sqrt{n}\rceil + 1 \geq r(C_4, K_{1,n}) \geq n + \sqrt{n} - 6n^{11/40}$$

[55]P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, On cycle–complete graph Ramsey numbers, *J. Graph Theory* **2** (1978), 53–64.

[56]N. Alon, Independence numbers of locally sparse graphs and a Ramsey type problem, *Random Structures and Algorithms* **9** (1996), 271-278.

[57]J. A. Bondy and P. Erdős, Ramsey numbers for cycles in graphs, *J. Combinatorial Theory Ser. B* **14** (1973), 46–54.

[58]S. A. Burr, P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, Some complete bipartite graph–tree Ramsey numbers, *Graph theory in memory of G. A. Dirac (Sandbjerg, 1985), Ann. Discrete Math., 41*, 79–89, North-Holland, Amsterdam-New York, 1989.

where the upper bound can be easily derived from the Turán number of C_4 and the lower bound can be found in[58]. Füredi can show (unpublished) that $r(C_4, K_{1,n}) = n + \lceil\sqrt{n}\rceil$ holds infinitely often.

A Ramsey problem for n-cubes (proposed by Burr and Erdős[37])
Let Q_n denote the n-cube on 2^n vertices and $n2^{n-1}$ edges. Prove that

$$r(Q_n) \leq c2^n.$$

The best known upper bound for $r(Q_n)$ is due to Beck [59] who showed that $r(Q_n) \leq c2^{n^2}$.

2.5. Multi-colored Ramsey numbers

For graphs G_i, $i = 1, \ldots, k$, let $r(G_1, \ldots, G_k)$ denote the smallest integer m satisfying the property that if the edges of the complete graph K_m are colored in k colors, then for some i, $1 \leq i \leq k$, there is a subgraph isomorphic to G_i with all edges in the i-th color. We denote $r(n_1, \ldots, n_k) = r(K_{n_1}, \ldots, K_{n_k})$. The only known exact value for a multi-colored Ramsey number is $r(3, 3, 3) = 17$ (see [11]). For $r(3, 3, 3, 3)$, the upper bound of 64 was established by Sanchez-Flores [60] in 1995 while the lower bound of 51 is about 25 years old [61]. Concerning $r(3, 3, 4)$, Piwakowski and Radziszowski [62] recently proved an upper bound of 29 while the lower bound of 28 is due to Kalbfleisch [63] and is more than 30 years old.

The multi-colored Ramsey numbers are related as follows (as a generalization of (2.5)):

$$r(k_1, k_2, \ldots, k_m) \leq 2 + \sum_{i=1}^{m}(r(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_m) - 1)$$

where strict inequality holds if $\sum_{i=1}^{m}(r(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_m) - 1)$ is even and for some i, $r(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_m)$ is even. Based on this fact, we

[59]J. Beck, An upper bound for diagonal Ramsey numbers, *Studia Sci. Math. Hungar.* **18** (1983), 401–406.

[60]A. Sanchez-Flores, An improved bound for Ramsey number $N(3, 3, 3, 3; 2)$, *Discrete Math.* **140** (1995), 281-286.

[61]F. R. K. Chung, On the Ramsey numbers N(3,3,. . . ,3;2), *Discrete Math.* **5** (1973), 317–321.

[62]K. Piwakowski and S. P. Radziszowski, New upper bound for the Ramsey number $R(3, 3, 4)$, preprint.

[63]J. G. Kalbfleisch, Chromatic graphs and Ramsey's theorem, Ph. D. thesis, University of Waterloo, Jan. 1966

can then derive:

$$\begin{aligned} r(\underbrace{3,\ldots,3}_{k})-1 &\leq 1+k(\ r(\underbrace{3,\ldots,3}_{k-1})-1) \\ &\leq k!(\frac{1}{k!}+\frac{1}{(k-1)!}+\ldots+\frac{1}{5!}+\frac{r(3,3,3,3)-1}{4!}) \\ &< k!(e-\frac{1}{12}) \end{aligned}$$

for $k \geq 4$.

The lower bound for $r(\underbrace{3,\ldots,3}_{k})$ is closely related to the Schur number s_k. A subset of numbers S is said to be *sum-free* if whenever i and j are (not necessarily distinct) numbers in S then $i+j$ is *not* in S. The Schur number s_k is the largest integer such that numbers from 1 to s_k can be partitioned into k sum-free sets. It can be shown [64] that, for $k \geq l$,

$$r(\underbrace{3,\ldots,3}_{k})-2 \geq s_k \geq c(2s_l+1)^{k/l}$$

for some constant c.

Using a result of Exoo [65] giving $s_5 \geq 160$, this implies

$$r(\underbrace{3,\ldots,3}_{k}) \geq c(321)^{k/5}.$$

Conjecture ($250, a very old problem of Erdős')

Determine

$$\lim_{k\to\infty}(r(\underbrace{3,\ldots,3}_{k}))^{1/k}.$$

It is known (see [61]) that $r(\underbrace{3,\ldots,3}_{k})$ is supermultiplicative in k so that the above limit exists.

Problem ($100)
Is the above limit finite or not?

Any improvement for small values of k will give a better general lower bound. The current range for this limit is between $(321)^{1/5} \approx 3.171765\ldots$ and infinity.

[64] F. R. K. Chung and C. M. Grinstead, A survey of bounds for classical Ramsey numbers, *Journal of Graph Theory* **7** (1983), 25-37.

[65] G. Exoo, A lower bound for Schur numbers and multicolor Ramsey numbers, *Electronic J. of Combinatorics* **1** (1994), # R8.

A conjecture on the ratio of multi-Ramsey numbers and Ramsey numbers (Proposed by Erdős and Sós[66])

$$\frac{r(3,3,n)}{r(3,n)} \to \infty$$

as $n \to \infty$.

Erdős[66] said, "It is very surprising that this problem which seems trivial at first sight should cause serious difficulties. We further expect that

$$\frac{r(3,3,n)}{n^2} \to \infty$$

as $n \to \infty$ and perhaps

$$r(3,3,n) > n^{3-\epsilon}$$

for every $\epsilon > 0$ if n is sufficiently large."

A multi-colored Ramsey problem for odd cycles (proposed by Erdős and Graham [36])

Show that for $n \geq 2$ and any k,

$$\lim_{k\to\infty} \frac{r(\overbrace{C_{2n+1},\ldots,C_{2n+1}}^{k})}{r(\underbrace{3,\ldots,3}_{k})} = 0$$

This problem is open even for $n = 2$.

A multi-colored Ramsey problem for even cycles (proposed by Erdős and Graham [36])
Determine $r(\underbrace{C_{2m},\ldots,C_{2m}}_{k})$.

It was proved [67] that

$$r(\underbrace{C_4,\ldots,C_4}_{k}) \leq k^2+k+1 \text{ for all } k$$
$$r(\underbrace{C_4,\ldots,C_4}_{k}) > k^2-k+1 \text{ for prime power } k.$$

The following upper and lower bounds for $r(\underbrace{C_{2m},\ldots,C_{2m}}_{k})$ were given in [36]:

[66] P. Erdős and V. T. Sós, Problems and results on Ramsey-Turán type theorems (preliminary report), *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979), Congress. Numer. XXVI*, pp. 17–23, Utilitas Math., Winnipeg, Man., 1980.

[67] F. R. K. Chung and R. L. Graham, On multicolor Ramsey numbers for complete bipartite graphs, *J. Comb. Th.* (B) 18 (1975), 164-69.

$$ck^{1+1/2m} \le r(\underbrace{C_{2m}, \ldots, C_{2m}}_{k}) \le c' k^{1+1/(m-1)}.$$

The lower bound can be further improved by using results in [68]:

$$r(\underbrace{C_{2m}, \ldots, C_{2m}}_{k}) \ge c"(\frac{k}{\log k})^{1+2/(3m-5)}.$$

A problem on three cycles
(proposed by Bondy and Erdős[36])
Show that

$$r(C_n, C_n, C_n) \le 4n - 3.$$

For odd n, if the above inequality is true, it is the best possible. Recently, Łuczak (personal communication) has shown that $r(C_n, C_n, C_n) \le 4n + o(n)$.

A coloring problem for trees
(proposed by Erdős and Graham [36])
Is it true for trees T_n on n vertices that

$$r(\underbrace{T_n, \ldots, T_n}_{k}) = kn + O(1)?$$

This would follow from the Erdős-Sós conjecture on trees.

A multi-colored Ramsey problem for bipartite graphs
(proposed by Chung, Erdős and Graham [67,36])
Determine $r(\underbrace{K_{s,t}, \ldots, K_{s,t}}_{k})$.

In [67], the following bounds are given:

$$(2\pi\sqrt{st})^{1/(s+t)}((s+t)/e^2)k^{(st-1)/(s+t)} \le r(\underbrace{K_{s,t}, \ldots, K_{s,t}}_{k}) \le (t-1)(k + k^{1/s})^s$$

for $k > 1, 2 \le s \le t$.

In particular, it would be of interest to determine $r(\underbrace{K_{3,3}, \ldots, K_{3,3}}_{k})$.

By using Turán numbers (to be discussed in the next chapter), we can show

$$r(\underbrace{K_{3,3}, \ldots, K_{3,3}}_{k}) > c\frac{k^3}{\log^3 k}.$$

[68] F. Lazebnik, V. A. Ustimenko and A. J. Woldar, A new series of dense graphs of high girth, *Bull. Amer. Math. Soc.* **32** (1995), 73–79.

For $r(G_1, G_2, \ldots, G_k)$, some exact results are known when $k \leq 3$ and the G_i's are cycles, and for the case that G_1 is a large cycle and the others G's are either odd cycles or complete subgraphs [55].

2.6. Size Ramsey numbers

The *size Ramsey number* $\hat{r}(G, H)$ is the least integer m for which there exists a graph F with m edges so that in any coloring of the edges of F in red and blue, there is always either a red copy of G or a blue copy of H. Sometimes we write $F \to (G, H)$ to denote this. For $G = H$, we denote $\hat{r}(G, G)$ by $\hat{r}(G)$.

A size Ramsey problem for bounded degree graphs
(proposed by Beck and Erdős [53])

For a graph G on n vertices with bounded degree d , prove that

$$\hat{r}(G) \leq cn$$

where c depends only on d.

The case for paths was proved by Beck [69](also see [70]) by using the following very nice result of Pósa [71]: Suppose that in a graph G, any subset X of the vertex set of size at most n satisfies:

$$|\{y \notin X : y \sim x \in X\}| \geq 2|X| - 1.$$

Then G contains a path with $3n - 2$ vertices.

Based on this result, Alon and Chung [72] explicitly construct a graph with cn edges so that no matter how we delete all but an ϵ-fraction of the vertices or edges, the remaining graph still contains a path of length n.

We point out that a directed version of this problem was considered by Erdős, Graham and Szemerédi[73] in 1975. Let $g(n)$ denote the least integer such that there is a directed acyclic graph G with $g(n)$ edges having the property that for any set X of n vertices of G, there is a directed path on G of length n which does not hit X. Then they show

$$c_1 \frac{n \log n}{\log \log n} < g(n) < c_2 n \log n$$

for constants $c_1, c_2 > 0$.

[69]J. Beck, On size Ramsey number of paths, trees, and circuits, I, *J. Graph Theory* **7** (1983), 115–129.

[70]J. Nešetřil and V. Rödl, eds., *Mathematics of Ramsey Theory*, Springer-Verlag, Berlin, 1990.

[71]L. Pósa, Hamiltonian circuits in random graphs, *Discrete Math.* **14** (1976), 359-364.

[72]N. Alon and F. R. K. Chung, Explicit constructions of linear-sized tolerant networks, *Discrete Math.* 72 (1988), 15-20.

[73]P. Erdős, R. L. Graham and E. Szemerédi, On sparse graphs with dense long paths, *Computers and mathematics with applications*, pp. 365–369, Pergamon, Oxford, 1976

Friedman and Pippenger [74] extended Pósa's result:

Suppose that in a graph G, any subset X consisting of at most $2n-2$ vertices satisfies:

$$|\{y \notin X : y \sim x \in X\}| \geq (d+1)|X|.$$

Then G contains every tree with n vertices and maximum degree at most d.

Using the above fact, they showed that

$$\hat{r}(T) \leq cn$$

for any tree with n vertices and bounded maximum degree.

Haxell, Kohayakawa, and Łuczak [75] proved that the size Ramsey number for C_n has a linear upper bound.

For the complete graph K_n, Erdős, Faudree, Rousseau and Schelp [50] proved that

$$\hat{r}(K_n) = \binom{r(n)}{2}.$$

They asked the following size Ramsey problem for $K_{n,n}$:

Problem Determine $\hat{r}(K_{n,n})$.

Erdős, Faudree, Rousseau and Schelp [76], and Nešetřil and Rödl [77] proved the following upper bound for $\hat{r}(K_{n,n})$.

$$\hat{r}(K_{n,n}) < \frac{3}{2}n^3 2^n.$$

For the lower bound, Erdős and Rousseau [78] proved by probabilistic methods that for $n \geq 6$,

$$\hat{r}(K_{n,n}) > \frac{1}{60}n^2 2^n.$$

[74] J. Friedman and N. Pippenger, Expanding graphs contain all small trees, *Combinatorica* **7** (1987), 71–76.

[75] P. E. Haxell, Y. Kohayakawa and T. Łuczak, The induced size-Ramsey number of cycles, *Combin. Probab. Comput.* **4** (1995), 217–239.

[76] P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, The size Ramsey number, *Period. Math. Hungar.* **9** (1978), 145–161.

[77] J. Nešetřil and V. Rödl, The structure of critical graphs, *Acta. Math. cad. Sci. Hungar.* **32** (1978), 295-300.

[78] P. Erdős and C. C. Rousseau The size Ramsey number of a complete bipartite graph, *Discrete Math.* **113** (1993), 259-262.

A size Ramsey problem
(proposed by Burr, Erdős, Faudree, Rousseau and Schelp [79])

For $F_1 = \cup_{i=1}^{s} K_{1,n_i}$ and $F_2 = \cup_{i=1}^{t} K_{1,m_i}$, prove that

$$\hat{r}(F_1, F_2) = \sum_{k=2}^{s+t} l_k$$

where $l_k = \max\{n_i + m_j - 1 : \ i + j = k\}$.

It was proved in [79] that

$$\hat{r}(sK_{1,n}, tK_{1,m}) = (m + n - 1)(s + t - 1).$$

The following problems are due to Erdős, Faudree, Rousseau and Schelp [79] [80].

A Ramsey size linear problem
(proposed by Erdős, Faudree, Rousseau and Schelp [80])

Suppose a graph G satisfies the property that every subgraph of G on p vertices has at most $2p - 3$ edges. Is it true that, for any graph H on n edges,

$$r(G, H) \leq cn? \qquad (2.18)$$

In [80], it was shown that for a graph G with p vertices and q edges, we have

$$r(G, K_n) > c(n/\log n)^{(q-1)/(p-2)}$$

for n sufficiently large.

This implies that for a graph G with p vertices and $2p-2$ edges, the inequality (2.18) does not hold for all H with n edges.

In the other direction, in [80] it was shown that for any graph G with p vertices and at most $p + 1$ edges, (2.18) holds.

In [80], Erdős, Faudree, Rousseau and Schelp raised the following problems:

For a graph G, where G is Q_3, $K_{3,3}$ or H_5 (formed by adding two vertex-disjoint chords to C_5), is it true that

$$r(G, H) \leq cn$$

for any graph H with n edges?

Suppose $r(G, T_n) \leq cn$ for any tree T_n on n vertices and $r(G, K_n) \leq cn^2$. Is it true that

$$r(G, H) \leq cn$$

for any graph H with n edges?

[79] S. A. Burr, P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, Ramsey-minimal graphs for multiple copies, *Nederl. Akad. Wetensch. Indag. Math.* **40** (1978), 187–195.

[80] P. Erdős, R. Faudree, C. C. Rousseau and R. H. Schelp, Ramsey size linear graphs, *Combin. Probab. Comput.* **2** (1993), 389–399.

What is the best constant c satisfying

$$r(C_{2k+1}, H) \leq c(2k+1)n$$

where H is any graph on n edges without isolated vertices?

Is it true that

$$r(C_m, H) \leq 2n + \lceil (m-1)/2 \rceil$$

where $m \geq 3$ and H is a graph consisting of n edges without isolated vertices?

2.7. Induced Ramsey numbers

The *induced Ramsey number* $r^*(G)$ is the least integer m for which there exists a graph H with m vertices so that in any 2-coloring of the edges of H, there is always an *induced* monochromatic copy of G in H. The existence of $r^*(G)$ was proved independently by Deuber [81], Erdős, Hajnal and Pósa [82], and Rödl [83]. It was proved by Harary, Neštřil and Rödl that [84] that that $r^*(P_4) = 8$. Erdős and Rödl [85] asked the following question:

Problem (proposed by Erdős and Rödl[49])
If G has n vertices, is it true that

$$r^*(G) < c^n$$

for some absolute constant c?

The above inequality holds for the case that G is a bipartite graph[83]. Łuczak and Rödl [86] showed that a graph on n vertices with bounded degree has its induced Ramsey number bounded by a polynomial in n, confirming a conjecture of Trotter.

Suppose G has k vertices and H has $t \geq k$ vertices. Kohayakawa, Prömel, and Rödl [87] proved that the induced Ramsey number $r^*(G, H)$ satisfies the following bound:

$$r^*(G, H) \leq t^{ck \log q}$$

---

[81]W. Deuber, Generalizations of Ramsey's theorem, *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I; Colloq. Math. Soc. János Bolyai, Vol. 10*, 323–332, North-Holland, Amsterdam, 1975.

[82]P. Erdős, A. Hajnal and L. Pósa, Strong embeddings of graphs into colored graphs, *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I; Colloq. Math. Soc. János Bolyai, Vol. 10*, 585–595, North-Holland, Amsterdam, 1975.

[83]V. Rödl, The dimension of a graph and generalized Ramsey theorems, thesis, Charles Univ. Praha, 1973.

[84]F. Harary, J. Neštřil and V. Rödl, Generalized Ramsey theory for graphs, XIV, Induced Ramsey numbers, *Graphs and other Combinatorial Topics* (Prague, 1982), 90-100.

[85]P. Erdős, Problems and results on finite and infinite graphs, *Recent advances in graph theory (Proc. Second Czechoslovak Sympos., Prague, 1974)*, 183–192 (loose errata), Academia, Prague, 1975.

[86]T. Łuczak and V. Rödl, On induced Ramsey numbers for graphs with bounded maximum degree, *J. Comb. Theory Ser. B* **66** (1996), 324–333.

[87]Y. Kohayakawa, H.-J. Prömel and V. Rödl, Induced Ramsey numbers, preprint.

where q denotes the chromatic number of H and c is some absolute constant. This implies

$$r^*(G) < k^{ck \log k}.$$

2.8. Ramsey theory for hypergraphs

A t-graph has a vertex set V and an edge set E consisting of some prescribed set of t-subsets of V. For t-graphs G_i, $i = 1, \ldots, k$, let $r_t(G_1, \ldots, G_k)$ denote the smallest integer m satisfying the property that if the edges of the complete t-graph on m vertices are colored in k colors, then for some i, $1 \le i \le k$, there is a subgraph isomorphic to G_i with all t-edges in the i-th color. We denote $r_t(n_1, \ldots, n_k) = r_t(K_{n_1}, \ldots, K_{n_k})$. Clearly, $r_2(n_1, \ldots, n_k) = r(n_1, \ldots, n_k)$.

The only known hypergraph Ramsey number is $r_3(4,4) = 13$, evaluated by direct computation[88]. Erdős, Hajnal and Rado[89] raised the following question:

Conjecture ($500)
(proposed by Erdős, Hajnal and Rado [89])

Is there an absolute constant $c > 0$ such that

$$\log \log r_3(n,n) \ge cn?$$

This is true if four colors are allowed [90].

If just three colors are allowed, there is some improvement due to Erdős and Hajnal (unpublished).

$$r_3(n,n,n) > e^{cn^2 \log^2 n}.$$

In [89], it was shown

$$2^{cn^2} < r_3(n,n) < 2^{2^n}. \tag{2.19}$$

Erdős[91] said, "We believe the upper bound is closer to the truth, although Hajnal and I [92] have a result which seems to favor the lower bound. We proved that if we color the triples of a set of n elements by two colors, there is always a set of size $s = \lfloor \sqrt{\log n} \rfloor$ on which the distribution is unbalanced, i.e., one of the colors contains at least $(\frac{1}{2} + \epsilon)\binom{s}{3}$ triples. This is in strong contrast to the case of $k = 2$, where it is possible to 2-color the pairs of an n-set so that in every set of size $f(n) \log n$,

[88]B. D. McKay and S. P. Radziszowski, The first classical Ramsey number for hypergraphs is computed, *Proceedings of the Second Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA'91, San Francisco, (1991), 304-308.

[89]P. Erdős, A. Hajnal and R. Rado, Partition relations for cardinal numbers, *Acta Math. cad. Sci. Hungar.* **16** (1965), 93–196.

[90]P. Erdős, A. Hajnal, A. Máté and R. Rado, *Combinatorial set theory: partition relations for cardinals, Studies in Logic and the Foundations of Mathematics, 106*, North-Holland Publishing Co., Amsterdam-New York, 1984.

[91]P. Erdős, Some of my favourite problems in number theory, combinatorics, and geometry, Combinatorics Week (Portuguese) (São Paulo, 1994), *Resenhas* **2** (1995), 165–186.

[92]P. Erdős and A. Hajnal, Ramsey-type theorems, Combinatorics and complexity (Chicago, IL, 1987), *Discrete Appl. Math.* **25** (1989) no. 1–2, 37–52

where $f(n) \to \infty$, both colors get asymptotically the same number of pairs. We would begin to doubt seriously that the upper bound in (2.19) is correct if we could prove that in any 2-coloring of the triples of an n-set, some set of size $s = (\log n)^\epsilon$ for which at least $(1-\eta)\binom{s}{3}$ triples have the same color. However, at the moment we can prove nothing like this."

Conjecture
(proposed by Erdős, Hajnal and Rado [89])
For every $t \geq 3$,

$$c \log_{t-1} n < r_t(n,n) < c' \log_{t-1} n$$

where $\log_u n$ denotes the u-fold iterated logarithm and c and c' depend only on t.

Generalized Ramsey problems

Denote by $F^{(t)}(n,\alpha)$ the largest integer for which it is possible to split the t-tuples of a set S of n elements into 2 classes so that for every $X \subset S$ with $|X| \geq F^{(t)}(n,\alpha)$, each class contains more than $\alpha\binom{|X|}{t}$ t-tuples of X . Note that $F^{(t)}(n,0)$ is just the usual Ramsey function $r_t(n,n)$. It is easy to show that for every $0 \leq \alpha \leq 1/2$,

$$c(\alpha)\log n < F^{(2)}(n,\alpha) < c'(\alpha)\log n.$$

Conjecture
(proposed by Erdős [74])
Prove that

$$F^{(2)}(n,\alpha) \sim c\log n$$

for an appropriate c and determine c.

As Erdös says in [93], the situation for $t \geq 3$ is much more mysterious. It is well-known[74] that if α is sufficiently close to 1/2, then

$$c_t(\alpha)(\log n)^{1/(t-1)} < F^{(t)}(n,\alpha) < c'_t(\alpha)(\log n)^{1/(t-1)}.$$

On the other hand, since $F^{(t)}(n,0)$ is just the usual Ramsey function, then the old conjecture of Erdős, Hajnal, Rado[89] would imply

$$c_1 \log_{t-1} n < F^{(t)}(n,0) < c_2 \log_{t-1} n.$$

Thus, assuming this conjecture holds, as α increases from 0 to 1/2, $F^{(t)}(n,\alpha)$ increases from $\log_{t-1} n$ to $(\log n)^{1/(t-1)}$.

Problem ($ 500)
Does the change in $F^{(t)}(n,\alpha)$ occur continuously, or are there jumps?

Erdős suspected there might only be one jump, this occurring at 0.

[93]P. Erdős, Problems and results on graphs and hypergraphs: similarities and differences, *Mathematics of Ramsey theory, Algorithms Combin., 5*, (J. Nešetřil and V. Rödl, eds.), 12–28, Springer, Berlin, 1990.