# A brief summary of the work of Fan Chung

## 1. Extremal graph theory

- *Universal graphs*
  The general problem of interest is: "What is the smallest graph which contains as subgraphs (or induced subgraphs) all members of a specified family of graphs?" For example, it was shown [113] that there is a graph on $O(n)$ vertices which contains all trees with bounded degrees on $n$ vertices as *induced* subgraphs. Furthermore, there is a graph on $O(n \log n)$ vertices and $O(n^2)$ edges which contains all planar graphs on $n$ vertices as *induced* subgraphs. Such problems on universal graphs have applications in data representations [69], parallel architectures and computation [96,108] and circuit design [80]. There are a number of results on determining the universal graphs which contain as subgraphs or induced subgraphs various families of graphs, such as trees [6,8,34,35,42], sparse graphs [43], caterpillars [49], planar graphs [113,116] and graphs with bounded arboricity [113].

- *Unavoidable graphs*
  A complementary problem of the universal graph is the following: "What is the largest subgraph that must occur in any graph with given number of vertices and edges?" In a paper with Erdős [61] sharp bounds on the size of the maximum unavoidable graphs are determined. In [62], the corresponding problem for the 3-uniform hypergraphs was solved. There are several improvements [90, 94] concerning these fundamental problems of unavoidable subconfigurations in specified families of combinatorial objects.

- *Graph decompositions*
  "For a given graph, find a partition (or covering) into the fewest possible subgraphs of a specified type." Such results were obtained for partitions into trees [16], bipartite graphs [23,29,56], or on fractional coverings of hypergraphs [103].

  A natural extension is the problem: "For two graphs, find a partition into mutually isomorphic pairs of subgraphs." In papers [38,50,51,74], the minimum number of subgraphs was determined for this simultaneous decomposition problem for graphs and hypergraphs.

  Graph decomposition problems have many applications in network design [18, 21], reliability and testing [95].

- *Ramsey theory*
  Ramsey theory basically concerns partitioning the complete graph into subgraphs each of which avoids certain graphs. The lower bound on the

classical Ramsey number $r(3,3,3,3;2)$ in [1] is still the best lower bound known so far. Among her papers [2,5,32,54,68,70,75,135,138] on Ramsey problems, [2,5] are on multi-colored Ramsey numbers with tight bounds for Ramsey numbers for four cycles, [54] settled a Ramsey problem of Erdős, and [32, 135] concern explicit constructions for Ramsey graphs. [138] contains several best bounds known on Ramsey-Turán problems on hypercubes.

## 2. Quasi-random graphs

The approach of quasi-random graphs is a unified framework for classifying various graph properties. In a series of papers [112,121,123,127,128,130, 131,132,135], a large family of graph properties, all of which are shared by random graphs, are shown to be equivalent in the sense that if a graph satisfies one of the properties, it must satisfy all of them. The list of equivalent graph properties includes some hard-to-compute properties such as the expansion property, the discrepancy property, subgraph enumeration properties as well as the neighborhood intersection property and the eigenvalue property (that are easy to compute). Thus, this provides a validation scheme for approximating one graph property by using other equivalent properties. The list of equivalent properties is still growing and each addition of new equivalent properties further strengthens the equivalence class as a whole.

In [123,130], we examine quasi-randomness for general families of hypergraphs and Boolean functions, which have additional structures and can be described as many related equivalence classes. Cohomological methods in [134] are useful for separating different classes. There is a natural relation between quasi-random classes of hypergraphs and the communication complexity of Boolean functions [132]. A similar approach can be used to analyze other combinatorial structures, such as sequences [135] and tournaments [127]. Many new directions and problems are open, especially on quantitative estimates of the relations between different graph properties. The theory of quasi-randomness has already been included in three books and in the Handbook of Combinatorics (also see Chapter 5 of [176]).

## 3. Diameters and routing

The diameter of a graph is the maximum distance among all pairs of vertices and it is a key invariant in communication networks. In a 1989 paper [111], an inequality between the diameter, a combinatorial invariant, and eigenvalues, an algebraic invariant, was established. This inequality can be used to derive isoperimetric inequalities between two subsets of vertices in a graph [160,161]. The higher-ordered eigenvalues $\lambda_k$ are related to isoperimetric inequalities involving $k$ distinct subsets of the vertices of a graph. The methods here can be used to obtain eigenvalue bounds for Riemannian manifolds [160] as well.

A closely related problem concerns finding a set of "short" paths joining specified pairs of vertices. Numerous problems in distributed and parallel computing involve routing messages or packets simultaneously along "short" paths with "small" congestion. It was shown in [149] (also see Chapter 4 of [176]) that the length of the paths and the number of time units for routing can both be estimated by eigenvalues.

In [64,105], the changes of the diameter are examined when edges are deleted or added. Sharp bounds are derived. A special interesting case is to add a random matching to a cycle or to an expander graph. In these cases, tight bounds of the same order as the optimal was obtained in [104]. There are several papers on related problems arising in communication networks, such as forwarding index in communication networks [78], capacity and through-put in packet switched networks [82, 87], optimizing blocking probabilities of switching networks [3,17,19,20], network designs [21,26,36,40,52], permutation networks and rearrangeable networks [4,31,85], nonheiarchical routing [52,79] and optical network design and codes [97].

**4. Spectral graph theory**

Most important properties of a graph are related to its eigenvalues. However, it is only recently that it has been possible to make this connection precise. In the new book *CBMS Lecture Notes on Spectral Graph Theory* [176] and a dozen or so papers [138,153,154,156,157,158,159,160,161,162,163,164, 169,170,171,172,173,175], new techniques have been developed to control many graph invariants in terms of eigenvalues and eigenfunctions. In particular, this involves a strong two-way interaction between concepts and methods from continuous mathematics and their emerging discrete counterparts.

For example, in [154] (also in Chapter 2 of [176]), a new characterization for discrete Cheeger constants was established. It can be used to obtain bounds for isoperimetric problems for cartesian products of graphs [169]. In [158,162] (see Chapter 8 of [176]), Dirichlet and Neumann eigenvalues for subgraph of a graph were considered. The Dirichlet eigenvalues are intimated related to generalizations and variations of the classical matrix-tree theorem of Kirchhoff which is effective for dealing with certain enumeration problems [162]. The Neumann eigenvalues are useful for dealing with random walk problems. By using a heat kernel inequality established in [158], eigenvalue lower bounds can be obtained for convex subgraphs (under certain density conditions). Many combinatorial enumeration problems (such as the contingency table problem and the knapsack problem) can be reduced to random walk problems on convex subgraphs of a homogeneous graph. Thus the eigenvalue lower bounds can be used to bound the rate of convergence of the random walks (see [158,159] and Chapter 10 of [176]) and polynomial approximation algorithms can be derived for these problems.

## 5. Discrete geometry

An old problem of Paul Erdős in combinatorial geometry is the following : Determine the maximum number of different distances among $n$ points in the plane. A lower bound of $n^{5/7}$ was given in [67] which improved the previous bound of Moser in 1952. This bound was subsequently improved and the current best bound of $n^{4/5}$ is obtained in [124]. Several techniques involving point-and-line incidence relations and point-and-sphere incidence relations among points in the plane and higher dimensional spaces are established in [110].

The Steiner tree problem is to find the shortest network connecting $n$ given points in the plane. There are several results [7,11,12,14,25,65,115] on Steiner tree problems in the plane, of bounded sets, or in higher dimensional spaces. The bounds [65] for the Euclidean space was later on improved by others. However, many problems for lattice points and for higher dimensional spaces still remain unsolved.

In a classical paper from 1935, Erdős and Szekeres showed that for each $n$ there exists a least value $g(n)$ such that any subset of $g(n)$ points in the plane in general position must always contain the vertices of a convex $n$-gon. The upper bounds for $g(n)$ that they established was recently improved in [175].

## 6. Graph embeddings and parallel computation

One of the classical problems on graph labeling is the bandwidth problem which is to arrange the vertices of a graph into a line (of integers) such that the maximum stretch of edges of the graph is minimized. The cutwidth problem is to arrange vertices in a line so that the maximum number of edges crossing the $i$-th place, over all $i$, is minimized. Both bandwidth and cutwidth problems are hard problems for a general graph. It is of interest to find a characterization for graphs with small bandwidth. In [109] we show that a density condition and a binary tree containment condition are necessary but not sufficient conditions for graphs with small bandwidth. However, these are necessary and sufficient conditions for graphs with small cutwidth and topological bandwidth.

In general we can consider the problem of embedding a graph $G$ into a host graph $H$ such that the dilation (i.e., the maximum stretch of edges of $G$ into $H$) and the congestion (i.e., the maximum frequency of edges of $H$ being used in paths associated with edges of $G$) are small. The bandwidth and cutwidth problems are special cases when the host graph is a path. Such graph embedding problems arising in network design and parallel computing concern the tradeoff of dilation and congestion [45,46,47,66,86]; comparisons of the host graphs [96,108], minimizing edge crossings [81], and the fault-tolerant problems of graph embeddings [80,96,114,116,146].