Discrete Applied Mathematics 37/38 (1992) 73-94
North-Holland

Graphs with small diameter after edge deletion

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Received 21 March 1990
Revised 11 February 1991

Abstract

Chung, F.R.K., Graphs with small diameter after edge deletion, Discrete Applied Mathematics 37/38 (1992) 73–94.

For given integers n and D, what is the minimum number of edges in a graph on n vertices with the property that after deleting any edge, the remaining graph has diameter no more than D? This problem was first proposed by Murty and Vijayan in 1964. In this paper, we give an exact solution for this problem for general n and D.

1. Introduction

Suppose a communication network has n centers and a certain number of two-way communication lines joining the centers. From each center, a message can be sent to any other center by traveling through at most D of the lines. Furthermore, the network satisfies the following reliability condition: if any s of the lines fail, then it is still possible to send information from each center to any other center by traveling through at most D' lines. The problem of interest is to determine the minimum number of lines needed to construct such a system. In other words, given integers n, D, D' and s, what is the minimum number $g(n,D,D',s)$ of edges in a graph G on n vertices with the property that G has diameter $\leq D$ and after deleting any s edges the remaining graph has diameter $\leq D'$?

This problem was first raised by Murty and Vijayan [22] in 1964. In the past, this problem has attracted the attention of many researchers (see Murty [17–19], Bollobás [1,2], Bondy and Murty [11], Bollobás and Eldridge [8], Caccetta [12–14] and Bollobás and Erdös [9]). The previous known results are for the cases where D and/or s are small, namely, ($D=2$, $s\geq 1$), ($D\leq 4$, $s=1$) or $D'=n-1$. For $D'=n-1$, the problem of determining $g(n,D,n-1,s)$ is just the problem of determining the minimum number of edges in a graph with connectivity $s+1$ and diameter D, which was solved by Bollobás [3,4]. A result in [16] states that after deleting s edges, the diameter can increase by at most a factor of $s+1$. Therefore, $g(n,D,D',s)=g(n,D,n-1,s)$ if $D'\geq(s+1)D$.

Since the general problem was believed to be very difficult, special attention was paid on the following problem:

For given integers n and D, what is the last number $g(n,D)$ of edges in a graph G with the property that after deleting any edge the remaining graph has diameter $\leq D$?

Upper bounds for $g(n,D)$ can be established by considering the following two types of constructions:

(i) For $D\equiv 0$ or 1 (mod 3), let G_1 be the graph formed by taking unions of cycles of length at most $\lfloor 2D/3\rfloor+1$ (except for one cycle of length at most $\lfloor 2D/3\rfloor+2$ if $D\equiv 1$ (mod 3)) and all cycles share exactly one special vertex (see Fig. 1(a)).

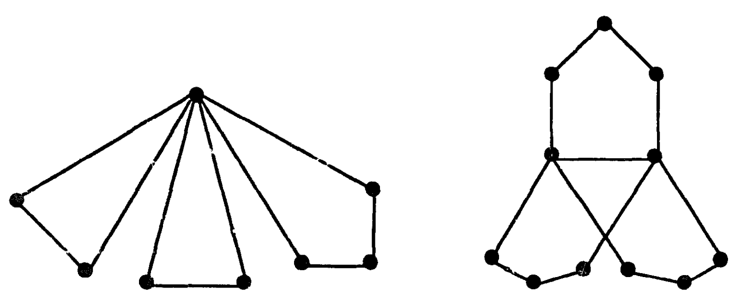
(ii) For $D\equiv 2$ (mod 3), let G_2 be the graph formed by joining two adjacent vertices by paths of at most $\lfloor 2D/3\rfloor+1$ edges except for one path of one edge (see Fig. 1(b)).

It is not too difficult to check that the above graphs give the upper bound

$$g(n,D)\leq n-1+\left\lceil\frac{n-1-\varepsilon}{\lfloor 2D/3\rfloor}\right\rceil$$

where $\varepsilon=1$ if $D\not\equiv 0$ (mod 3) and $\varepsilon=0$ otherwise.

In this paper, we will prove that any graph (multiple edges allowed) on n vertices which satisfies the property that after deleting any edge the remaining graph has diameter $\leq D$ must indeed contain at least $n-1+\lceil(n-1-\varepsilon)/\lfloor 2D/3\rfloor\rceil$ edges if $n>\lfloor 3D/2\rfloor+1$. This is a somewhat stronger version of the conjecture in [10]. For $n\leq D+1$, $g(n,D,1)=n$ and the minimum graph is a cycle on n vertices. For $D+1<n\leq\lfloor 3D/2\rfloor+1$, $g(n,D,1)=n+1$ and a minimum graph is a union of three paths of length (i.e., the number of edges) at most $\lceil D/2\rceil$, $\lfloor D/2\rfloor+1$, respectively, joining two vertices.



n = 8
D = 4

(a)

n = 11
D = 5

(b)

Fig. 1.

In the next section, we will prove some useful facts and consider some special cases. In Section 5, we will first prove a slightly weaker result:

$$g(n,D) \geq n + \frac{n-1-\varepsilon}{\lfloor 2D/3 \rfloor} - 2 \quad \text{for } n > \lfloor 3D/2 \rfloor + 1.$$

And finally in Section 6, further arguments will be provided to prove the following main theorem.

Main Theorem.

$$g(n,D) = \begin{cases} n, & \text{if } n \leq D+1, \\ n+1, & \text{if } D+1 < n \leq \lfloor 3D/2 \rfloor + 1, \\ n-1+\lceil (n-1-\varepsilon)/\lfloor 2D/3 \rfloor \rceil, & \text{if } n > \lfloor 3D/2 \rfloor + 1, \end{cases}$$

where $\varepsilon = 1$ *if* $D \not\equiv 0 \pmod 3$ *and* $\varepsilon = 0$ *if* $D \equiv 0 \pmod 3$.

2. Useful facts and the cases for small n

Since the case of $D \leq 4$ was proved in [13], we may assume $D > 4$. The proof is by induction on n and we assume it holds for any graph with fewer than n vertices. Suppose G has n vertices with $g(n,D)$ edges satisfying the property that after deleting any edge E the remaining graph, denoted by $G-E$ (with vertex set $V(G)$ and edge set $E(G)-\{E\}$), has diameter no more than D. We will first state a few useful facts some of which can be easily proved and we will not include the proofs here.

Fact 2.1. *G is 2-edge-connected.*

Fact 2.2. $g(n,D) \geq n$.

Fact 2.3. $g(n,D) \geq n+1$ *if* $n > D+1$.

Proof. The unique 2-edge-connected graph on n vertices and n edges is a cycle. For $n > D+1$, G cannot be a cycle since $G-E$ has diameter $\leq D$ for any edge E. Therefore $g(n,D)$ must be at least $n+1$. □

Fact 2.4. $g(n,D) = n+1$ *if* $D+1 < n \leq \lfloor 3D/2 \rfloor + 1$.

Proof. We consider a graph on n vertices consisting of three paths joining two vertices. The number of edges in the three paths are at most $\lceil D/2 \rceil$, $\lfloor D/2 \rfloor + 1$ and $\lfloor D/2 \rfloor + 1$, respectively. It can be easily verified that the graph has diameter D after any edge is deleted. Therefore $g(n,D) \leq n+1$ if $D+1 < n \leq \lfloor 3D/2 \rfloor + 1$. □

Fact 2.5. *$g(n, D) > n+1$ if $n > \lfloor 3D/2 \rfloor + 1$.*

Proof. Suppose $n > \lfloor 3D/2 \rfloor + 1$ and G has $n+1$ edges. Since G is 2-edge-connected on n vertices and $n+1$ edges, G must be either (a) a graph G_1 consisting of three paths, with disjoint internal vertices joining two vertices, or (b) a graph G_2 consisting of two cycles with just one common vertex. If (a) holds, suppose three paths have a, b, c edges, respectively, where $a \ge b \ge c$. Because of the edge-deletion property, $a-1+\lfloor (b+c)/2 \rfloor \le D$. Subject to $a \ge b \ge c$, $a+b+c$ is maximized when $a = \lfloor D/2 \rfloor + 1$, $b = \lfloor D/2 \rfloor + 1$ and $c = \lceil D/2 \rceil$. The number of vertices in G_1 is $a+b+c-1 \le \lfloor 3D/2 \rfloor + 1$, a contradiction.

If (b) is true, suppose the two cycles have x and y edges, respectively, where $x \ge y$. Because of the edge-deletion property, $x - 1 + \lfloor y/2 \rfloor \le D$. Subject to the inequality $x \ge y$, $x+y$ is maximized when $x = \lceil 2D/3 \rceil + 1$ and $y = \lfloor 2D/3 \rfloor + 1$. The number of vertices in G_2 is $x+y-1 \le \lfloor 4D/3 \rfloor + 1 \le \lfloor 3D/2 \rfloor + 1$ for $D > 4$, again a contradiction.

Therefore we conclude that G has more than $n+1$ edges if $n > \lfloor 3D/2 \rfloor + 1$. □

From now on we will assume $n > \lfloor 3D/2 \rfloor + 1$. The following fact was first pointed out by Leighton [20].

Fact 2.6. *If G contains a cycle of length $\le \lfloor 2D/3 \rfloor + 1$, and $n > \lfloor 3D/2 \rfloor + 1$, then G has at least $n - 1 + (n-1-\varepsilon)/\lfloor 2D/3 \rfloor$ edges.*

Proof. Suppose G contains a cycle C of length $\le \lfloor 2D/3 \rfloor + 1$. We consider G' by contracting all vertices in C into one single vertex and delete edges in C. Clearly, $n' = |V(G')| = n - |C| + 1$ and $e' = |E(G')| = e - |C|$. Suppose $n' > \lfloor 3D/2 \rfloor + 1$. Since G' has the edge-deletion property, by induction we have

$$e' \ge n' - 1 + \frac{n'-1-\varepsilon}{\lfloor 2D/3 \rfloor},$$

i.e.,

$$e \ge n + \frac{n - |C| - \varepsilon}{\lfloor 2D/3 \rfloor} \ge n - 1 + \frac{n-1-\varepsilon}{\lfloor 2D/3 \rfloor}.$$

We may assume $n' \le \lfloor 3D/2 \rfloor + 1$. Therefore $n \le \lfloor 3D/2 \rfloor + \lfloor 2D/3 \rfloor + 1$ and $(n-1-\varepsilon)/\lfloor 2D/3 \rfloor < 4$ since $D > 4$. If G has fewer than $n - 1 + \lceil (n-1-\varepsilon)/\lfloor 2D/3 \rfloor \rceil$ edges, G has at most $n+1$ edges. The proof of Fact 2.5 implies that G has at most $\lfloor 3D/2 \rfloor + 1$ vertices. This contradicts our assumption that $n > \lfloor 3D/2 \rfloor + 1$ and the fact is proved. □

From now on we will assume that any cycle in G must have more than $\lfloor 2D/3 \rfloor + 1$ edges.

3. A weighting function

A vertex is called a fat vertex if its degree is at least 3 in G. A path in G is called a *segment* if both endpoints are fat and the interior vertices are not. The endpoints of a segment may be identical.

Let S^* denote a segment with the maximum length having endpoints a and b. Without loss of generality, we assume degree $a \geq$ degree b. Let a' denote the neighbor of a in S^*. We now consider $G-E^*$ where $E^*=\{a,a'\}$ and we form the breadth-first search tree T from a'. It is easy to see that each edge E in T is either in a segment entirely lying in T or E is in a partial segment with one endpoint fat and the other a leaf in T. Such a partial segment will be called a leaf-segment. Here we will allow leaf-segments of zero length to make sure the number x of leaf-segments (or leaves) is exactly twice that of the edges in $E(G-E^*)-E(T)$. A leaf-segment can be denoted by v_E, where E is an edge in $E(G-S^*)-E(T)$ and v is an endpoint of E. A segment S either contains no leaf-segment or contains two leaf-segments.

Our goal is to show

$$W=(e-n+1)-\frac{n-1-\varepsilon}{L}\geq 0,$$

where L denotes $\lfloor 2D/3 \rfloor$ and e denotes the number of edges in G. Since $|E(T)|=n-1$, we have

$$2LW=2L+2\varepsilon+Lx-2\,|E(T)|$$

where x is the number of leaves in T, and $2(e-n)=x$.

We now let T^* be a subtree of T consisting of all edges in T but not in S^*. Clearly, $|E(T^*)|=|E(T)|-|E(S^*)|+1$. For each edge E in T^*, we now define a weighting function α. Let $z(E)$ denote the number of leaves u so that E is contained in the shortest paths joining u to b in T denoted by $P(u)$. We define

$$\alpha(E,u)=\begin{cases} 2/z(E), & \text{if } E\in P(u),\\ 0, & \text{otherwise.}\end{cases}$$

Since $\sum_u \alpha(E,u)=2$ for each E, we have the following:

$$\begin{aligned}2LW&\geq 2L-2\,|E(S^*)|+2+2\varepsilon+Lx-2\,|E(T^*)|\\ &\geq 2L-2\,|E(S^*)|+2+2\varepsilon+\sum_u\Big(L-\sum_{E\in P(u)}\alpha(E,u)\Big).\end{aligned}$$

Since each leaf u can be paired with its mate u^*, the above inequality can be deduced to the following:

$$\begin{aligned}2LW&\geq 2L-2\,|E(S^*)|+2+2\varepsilon\\ &\quad+\sum_{u,u^*}\Big(2L-\sum_{E\in P(u)}\alpha(E,u)-\sum_{E\in P(u^*)}\alpha(E,u)\Big).\end{aligned}\tag{1}$$

We now consider the following two cases:

Case 1: $|E(S^*)|=L+1+\varepsilon+t,\ t\geq 0$.

Let $|P(u)|$ denote the number of edges in $P(u)$. For each leaf u, we get

$$|P(u)|\leq D-(|E(S^*)|-1)=D-L-\varepsilon-t$$

since G has diameter at most D in $G-E^*$.

$$\sum_{E\in P(u)}\alpha(E,u)+\sum_{E\in P(u^*)}\alpha(E,u^*)\leq 4(D-L-\varepsilon-t)\leq 2L-4t.$$

Therefore $2LW\geq -2t+\sum_{u,u^*}4t\geq 0$.

It remains to prove the following case:

Case 2: $|E(S^*)|=L-t+\varepsilon$ *for* $L+\varepsilon>t\geq 0$.

The proof for Case 2 will be given in Section 5 after more facts are examined.

4. More facts

In this section we assume $|E(S^*)|=L-t+\varepsilon$ and $t\geq 0$. Again we choose S^* to be a segment of the maximum length as in the previous sections. From Fact 2.6 we know that the endpoints of S^* are different. Among all segments achieving the maximum length we will choose S^* to be the segment such that its endpoints, denoted by a and b, have maximum distance in the graph formed by deleting all interior vertices of S^*. Let $d_G(u,v)$ denote the distance of u and v in G.

We now consider a breadth-first search tree $\hat{T}$ growing from S^* in G. Namely, initially we set $\hat{T}_0=S^*$. For each i, $V(\hat{T}_{i+1})$ is $V(\hat{T}_i)$ together with all vertices of G that are adjacent to some vertices in T_i. For each vertex v in $V(\hat{T}_{i+1})-V(\hat{T}_i)$, we select exactly one edge joining v to some vertex in $V(\hat{T}_i)$ and place it in $E(\hat{T}_{i+1})$. A vertex v in $V(\hat{T}_{i+1})-V(\hat{T}_i)$ is said to be of rank $i+1$, denoted by $r(v)=i+1$. Also the edge in $E(T_{i+1})$ from v to $V(\hat{T}_i)$ is said to have rank $i+1$.

To prove $W=e-n+1-(n-1-\varepsilon)/L\geq 0$, it suffices to show

$$\begin{aligned}2LW&=2(e-n+1)L-2(n-1-\varepsilon)\\&=2\varepsilon-2\,|E(S^*)|+Ly-2\,|E(F)|\\&=2t-2L+Ly-2\,|E(F)|\geq 0\end{aligned}\tag{2}$$

where F denotes the forest formed by removing edges (and interior vertices) of S^* from $\hat{T}$ (F is the union of two trees T_a and T_b, containing a, b, respectively) and $y=2(e-n+1)$ denotes the number of leaves in F.

For a vertex v in F, $P(v)$ denotes the unique path in F joining v to a or b. Clearly, the number of edges in $P(v)$, denoted by $|P(v)|$, is $r(v)$. Let E_v denote the edge containing v in F with $r(E_v)=r(v)$. Also, let v_E denote the endpoint of E with $r(v)=r(E)$ (which is larger than the rank of the other endpoint of E). $P(E)$ denotes the unique path $P(v_E)$. In general, $P(u,v)$ denotes a shortest path joining u and v (breaking ties by choosing the one containing the maximum number of edges in $\hat{T}$.)

For a leaf u and its mate u^*, let $\bar{u}$ denote $\{u, u^*\}$, so called the leaf-dual of u. We denote $r(\bar{u}) = r(u) + r(u^*)$ and $P(\bar{u}) = P(u) \cup P(u^*)$. A leaf-dual $\bar{u}$ is said to be *short* if $r(\bar{u}) \leq L + t - \varepsilon + 1$. For each edge E, let $Z(E)$ denote the set of leaf-duals $\bar{u}$ with $P(\bar{u})$ containing E. An edge is said to be good if $Z(E)$ contains at least two short leaf-duals. E is called $\bar{u}$-special if $Z(E)$ contains exactly one short leaf-dual $\bar{u}$ and $|Z(E)| > 1$.

Fact 4.1. *$r(\bar{u}) \leq 2(D - L + t - \varepsilon + 1)$ and each vertex v is in a cycle (not necessarily simple) of length at most $2(D - L + t - \varepsilon) + 3$ which contains a. Analogously, each vertex v is in a cycle of length at most $2(D - L + t - \varepsilon) + 3$ containing b.*

Proof. For any vertex u, $d_{G-S^*}(u, a) + |E(S^*)| - 1 \leq D$, and $d_{G-S^*}(u, b) + |E(S^*)| - 1 \leq D$. Furthermore, there is a neighbor v of u in G not contained in $P(a, u)$. Either $d_{G-S^*}(a, v) \leq d_{G-S^*}(a, u)$ or u is in $P(a, v)$. If $d_{G-S^*}(a, v) \leq d_{G-S^*}(a, u)$, we obtain the cycle $P(a, u) \cup P(a, v) \cup \{u, v\}$ with length at most $2(D - L + t - \varepsilon) + 3$. If u is in $P(a, v)$, we consider a neighbor v' of v not in $P(a, v)$ and iterate this process. After a finite number of steps this process will stop and we have the desired cycle. □

Fact 4.2. *For a segment S with endpoints u and v in a graph G' with diameter D, every vertex w in G' satisfies*

$$|E(S)| + d_{G'}(u, w) + d_{G'}(v, w) \leq 2D + 1.$$

Proof. If $d_{G'}(u, w) + |E(S)| \leq D + 1$, then the fact holds. We may assume $D + 1 - d_{G'}(u, w) < |E(S)|$. Choose a vertex z in S with $d_S(z, u) = D + 1 - d_{G'}(u, w)$. Since G' has diameter D, the shortest path from z to w is through v and $d(z, w) = d_S(z, v) + d_{G'}(v, w) \leq D$. Since $d_S(z, v) = |E(S)| - d_S(z, u) = |E(S)| + d_{G'}(u, w) - D - 1$, we have

$$|E(S)| + d_{G'}(u, w) + d_{G'}(v, w) \leq 2D + 1. \quad \square$$

Fact 4.3 is an immediate consequence of Fact 4.2.

Fact 4.3. *For each edge E,*

$$2D - L - \varepsilon + t + 1 \geq d_{G-E}(v_E, a) + d_{G-E}(v_E, b).$$

Since the shortest path joining v_E and a in $G - E$ (and the shortest path joining v_E and b in $G - E$, respectively) contains leaves of F, the above inequality implies $2D - L - \varepsilon + t - 1 + 2r(E) \geq r(\bar{u}_1) + r(\bar{u}_2)$ for leaves u_1, u_2 in $Z(E)$ (not necessarily distinct).

Fact 4.4. *$Z(E)$ contains a short leaf-dual if*

$$2r(E) \leq 3L - 2D - \varepsilon + t + 4.$$

Proof. Let u denote the leaf in $Z(E)$ with the minimum rank. Then

$$d_{G-E}(v_E, a) \geq 1 + r(\bar{u}) - r(E).$$

Therefore, by Fact 4.3 we have

$$\begin{aligned} 2r(\bar{u}) &\leq 2D - L - \varepsilon + t - 1 + 2r(E) \\ &\leq 2L + 2t - 2\varepsilon + 3. \end{aligned}$$

This implies $\bar{u}$ is short. □

Fact 4.5. *Suppose an edge E satisfies the property that every $\bar{u} \in Z(E)$ has rank $r(\bar{u})$ at least $L + t - \varepsilon + 1 + s$. Then $r(E) \geq (3L - 2D + t + 3 - \varepsilon)/2 + s$.*

Proof. From Fact 4.3 we know

$$2D - L - \varepsilon + t - 1 + 2r(E) \geq 2(L + t - \varepsilon + 1 + s).$$

This implies

$$2r(E) \geq 3L - 2D + t + 2s - \varepsilon + 3$$

and

$$r(E) \geq \frac{t + 3 + 3L - 2D}{2} + s. \quad \square$$

Fact 4.6. *Suppose $2r(E) \leq 3L - 2D - \varepsilon + t + 4$ and E is not good. Let $E' \in P(E)$ and E' is not good. Then $Z(E') = Z(E)$.*

Proof. Clearly $Z(E) \subseteq Z(E')$. If there is a leaf u in $Z(E') - Z(E)$, let E'' denote the edge with the minimum rank in $P(E, E') - P(u)$. (Recall that $P(E, E')$ denotes the shortest path in F containing E and E'.) Let E''' be the edge with the minimum rank in $P(u, E') - P(E, E')$. Both E'' and E''' have rank no more than $r(E)$ and both are not good since E' is not good. By Fact 4.4, we know there is a short leaf in $Z(E'')$ and there is a short leaf in $Z(E''')$. This implies E' is good, a contradiction. Therefore we have $Z(E) = Z(E')$ as desired. □

Let S_1 denote the shortest path joining a and b in $G - E(S^*)$. It is easy to see that S_1 contains exactly one pair of leaves $\{u_1, u_1^*\}$. From Fact 2.6, we know that $r(\bar{u}_1) = |E(S_1)| - 1 = t + w - \varepsilon > t - \varepsilon$ for some $w > 0$.

Fact 4.7. $w \leq D - L$.

Proof. Recall that E^* is the edge $\{a, a'\}$ in S^* and we consider the distance between a and a' in $G - E^*$. We have

$$|S_1| + |S^*| - 1 \leq D.$$

This implies $w \le D-L$. □

Fact 4.8. *There are leaves u, u^* other than u_1, u_1^* satisfying $r(\bar{u}) \le 2D-2L+t-w-\varepsilon+1$.*

Proof. Choose an edge $E_1=\{a,v_1\}$ not in S_1. In $G'=G-E_1$, if $d_{G'}(v_1,b)=d_{G'}(a,b)$, there are leaves $\{u_2,u_2^*\}$ on the path joining v_1 and b different from $\{u_1,u_1^*\}$ and from Fact 4.1 we have

$$r(\bar{u}_1)+r(\bar{u}_2)+2 \le 2(D-L+t-\varepsilon)+3.$$

This implies

$$r(\bar{u}_2) \le 2D-2L+t-\varepsilon-w+1.$$

If $d_{G'}(v_1,b)=1+d_{G'}(a,b)$, we consider the edge $E_2=\{v_1,v_2\}$, where $r(v_2)>r(v_1)$, and ask if $d_{G'}(v_1,b)=d_{G'}(v_2,b)$ as before. This procedure must stop after a finite number of steps and we will then have leaves u, u^* with the required property. □

5. Another weighting function

We now consider a weighting function λ, which is defined for an edge E in T and a leaf-dual $\bar{u}$, where $u \in Z(E)$:

(i) If $|Z(E)|=1$ and $u \in Z(E)$, then we define $\lambda(E,\bar{u})=2$.

(ii) Suppose $|Z(E)|>1$. If E is not $\bar{u}$-special, we define

$$\lambda(E,\bar{u})=\begin{cases} 1, & \text{if } 2r(E)>r(\bar{u})-(L+t-\varepsilon),\\ \frac{1}{2}, & \text{if } r(E)=\dfrac{r(\bar{u})-(L+t-\varepsilon)}{2},\\ 0, & \text{otherwise.}\end{cases}$$

If E is $\bar{u}$-special, we define

$$\lambda(E,\bar{u})=\begin{cases} 2, & \text{if } 2r(E)<\min_{\bar{u}'\in Z(E)-\bar{u}} r(\bar{u}')-(L+t-\varepsilon),\\ \frac{3}{2}, & \text{if } r(E)=\dfrac{\min_{\bar{u}'\in Z(E)-\bar{u}} r(\bar{u}')-(L+t-\varepsilon)}{2},\\ 1, & \text{otherwise.}\end{cases}$$

We want to show that the definition of λ leads to the following useful facts. For convenience, we say an edge is $\bar{u}$-admissible if $\lambda(E,\bar{u})>1$ and E is $\bar{u}$-special.

Lemma 5.1. *For each E, $\sum_u \lambda(E,\bar{u}) \ge 2$.*

Proof. If $|Z(E)|=1$, we have $\sum_u \lambda(E,\bar{u}) \ge 2$. We may assume $|Z(E)|>1$. If E is good, we have $\sum_u \lambda(E,u) \ge 2$.

If E is $\bar{u}$-special for some $\bar{u}$, by definition $\sum_u \lambda(E,\bar{u}) \geq 2$ if $2r(E) < \min_{\bar{u}' \in Z(E)-\bar{u}} r(\bar{u}') - (L+t-\varepsilon)$ or $r(E) = (\min_{\bar{u}' \in Z(E)-\bar{u}} r(\bar{u}') - (L+t-\varepsilon))/2$. When $2r(E) > \min_{\bar{u}' \in Z(E)-\bar{u}} r(\bar{u}') - (L+t-\varepsilon)$, at least two leaf-duals contribute 1 each to $\sum_{\bar{u}} \lambda(E,\bar{u})$.

If E is not good and E is not $\bar{u}$-special for any u, then by Fact 4.4 we have $r(E) > (3L-2D-\varepsilon+t+4)/2$. Therefore if

$$2r(E) > r(\bar{u}) - (L+t-\varepsilon) \quad \text{for all } \bar{u} \text{ in } Z(E),$$

each $\bar{u} \in Z(E)$ contributes 1 to $\sum_u \lambda(E,\bar{u})$ and therefore $\sum_u \lambda(E,\bar{u}) \geq 2$. Suppose $2r(E) \leq r(\bar{u}) - (L+t-\varepsilon)$. From Fact 4.3, we have

$$\min\{d_{G-E}(v_E,a), d_{G-E}(v_E,b)\} + \left\lfloor \frac{L-t+\varepsilon}{2} \right\rfloor \leq D.$$

We have

$$r(\bar{u}) + 1 - r(E) + \left\lfloor \frac{L-t+\varepsilon}{2} \right\rfloor \leq D$$

or

$$\begin{aligned} \frac{3L-2D-\varepsilon+t+5}{2} + L+t-\varepsilon \leq r(E)+L+t-\varepsilon \leq r(\bar{u}) - r(E) \\ \leq D-1-\left\lfloor \frac{L-t+\varepsilon}{2} \right\rfloor. \end{aligned}$$

This is impossible. This completes the proof of the lemma. □

Lemma 5.1 implies the following

$$\sum_{\bar{u}} \sum_{E \in P(\bar{u})} \lambda(E,\bar{u}) \geq 2\,|E(F)|.$$

From (2) we can write

$$\begin{aligned} 2LW &= 2(e-n+1)L - 2(n-1-\varepsilon) \\ &= 2t - 2L + Ly - 2\,|E(F)| \\ &\geq 2t - 2L + \sum_{\bar{u}} \Big(2L - \sum_{E \in P(\bar{u})} \lambda(E,\bar{u})\Big). \end{aligned} \tag{3}$$

To prove the main theorem, it suffices to show the following:

$$\sum_{\bar{u}} \Big(2L - \sum_{E \in P(\bar{u})} \lambda(E,\bar{u})\Big) \geq 2L - 2t. \tag{4}$$

Lemma 5.2. *For any $\bar{u}$ which is not short we have*

$$2L \geq \sum_{E \in P(\bar{u})} \lambda(E,\bar{u}).$$

Proof. It follows from the definition of λ that

$$\sum_{E\in P(\bar u)} \lambda(E,\bar u)\le |\{E\in P(\bar u)\colon |Z(E)|=1\}| + |\{E\in P(\bar u)\colon 2r(E)>r(\bar u)-(L+t-\varepsilon)\}| + \tfrac12 \left|\left\{E\in P(\bar u)\colon r(E)=\frac{r(\bar u)-(L+t-\varepsilon)}{2}\right\}\right| \le |\{E\in P(\bar u)\colon |Z(E)|=1\}|+L+t-\varepsilon+1.$$

Since the longest segment is of length $L-t+\varepsilon$, i.e., $L-t+\varepsilon-1\ge|\{E\in P(\bar u)\colon |Z(E)|=1\}|$, we therefore have

$$\sum_{E\in P(\bar u)} \lambda(E,\bar u)\le 2L. \qquad \square$$

Lemma 5.3. *For any short $\bar u$, we have*

$$2L\ge \sum_{E\in P(\bar u)} \lambda(E,\bar u).$$

Proof. The lemma is obviously true for $\bar u$ if there are no $\bar u$-admissible edges. Suppose there are some $\bar u$-admissible edges.

Let W denote the set of $\bar u$-admissible edges. From Fact 4.6, we know $W_1 = W\cap P(u)$ consists of consecutive edges in $P(u)$ so does $W_2=W\cap P(u^*)$. Let x_i denote the minimum rank for edges in W_i, for $i=1,2$. We also assume $a\in P(u)$.

Let E be the edge in $P(u)$ with $r(E)=x_1$. Applying Fact 4.3, we get $2D-L-\varepsilon+t-1+2x_1\ge r(\bar u_1)+r(\bar u_2)$ for $\bar u_1,\bar u_2\in Z(E)$, while $\bar u_1$ and $\bar u_2$ are on the shortest paths joining v_E to a and b, respectively in $G-E$. Let r_1 denote the minimum rank of all nonshort leaf-duals in $Z(E)$. If $\bar u_1\ne\bar u_2$, we get

$$2D-L-\varepsilon+t-1+2x_1-r_1\ge r(\bar u). \tag{5}$$

We consider two possibilities.

Case (a): (5) *holds and suppose* $\sum_{E\in P(\bar u)}\lambda(E,\bar u)>2L$. We denote $r_1=L+t-\varepsilon+1+s$ for some $s\ge1$. (5) implies $r(\bar u)\le 2D-2L-2+2x_1-s$. Since $x_1\le(s+1)/2$, we have $r(\bar u)\le L+1$. In fact, we have $r(\bar u)\le L$ except for the case $x_1=(s+1)/2$, $\varepsilon=1$ and $D\equiv1 \pmod 3$. Suppose $\varepsilon=1$, $x_1=(s+1)/2$, $D\equiv1 \pmod 3$ and $r(\bar u)=L+1$. Since $\bar u$ is short, we have $r(\bar u)\le L+t-\varepsilon+1$ and $t\ge\varepsilon$. Since $\sum_{E:\,\bar u\text{-special}}(\lambda(E,\bar u)-1)\le1$, we have

$$\sum_E \lambda(E,\bar u)\le r(\bar u)+|\{E: |Z(E)|=1\}|+1$$

which is bounded above by $2L$ unless $t\le1$ and $|\{E:|Z(E)|=1\}|=L-t+\varepsilon-1$. From Fact 4.8 and $x_1=(s+1)/2\le(t+3)/2\le2$, we have $L+1\le 2D-2L+t-w-\varepsilon+1$. This implies $w\le t+1\le2$. Since the segment S' containing $\bar u$ has length $L-t+\varepsilon$, the endpoints of S' are connected by a path in $G-S'$ of length at most $t+w-\varepsilon+1\leq 3$. This implies $r_1\leq L+1-(L-t+\varepsilon-1)+(t+w-\varepsilon)\leq 2t+w-2\varepsilon+2\leq 4$. On the other hand, $r_1\geq L+t-\varepsilon+2\geq L+2$. This is impossible, since $D>4$ and $L\geq 3$. Therefore we have $\sum_E \lambda(E,\bar{u})\leq 2L$, as desired.

Case (b): *Suppose* (5) *does hot hold.* We may assume $\bar{u}_1=\bar{u}_2=\bar{u}$. We consider two cases.

Case (b.1): $W_2\neq\emptyset$. Let E' be the edge in W_2 with the minimum rank x_2. Also let c denote either a or b which is not in $P(u^*)$. We consider the following two possibilities.

Subcase (b.1.1): E' is not in the shortest path P joining v_E to c in $G-E$. There is a leaf-dual $\bar{u}''$ in $P-P(\bar{u})$ in $Z(E')$ and $\bar{u}''$ is not short. Therefore

$$2L+t-1+2x_1-r_2\geq r(\bar{u})$$

where

$$r(\bar{u}'')\geq r_2=\min_{v\in Z(E')-\bar{u}} r(\bar{v}).$$

This case can be proved by using similar arguments as in Case (a).

Subcase (b.1.2): E' is in P. Since the shortest path between a and b is at least $t+w-\varepsilon+1$, we have $|(P-P(\bar{u}))\cup(P(u^*)-P)|\geq t+w-\varepsilon+1$ and $w\geq 1$. From Fact 4.2 we have

$$2D+1\geq 2(r(\bar{u})+1-x_1-(x_2-1))+t+w-\varepsilon+1+L-t+\varepsilon.$$

That is

$$2D-L-4-w+2x_1+2x_2\geq 2r(\bar{u}).$$

Set $s_i=r_i-(L+t-\varepsilon+1)$. From Fact 4.5, we have

$$|\{E:|Z(E)|=1\}|\leq r(\bar{u})-\sum_{i=1,2}\left(s_i+\frac{3L-2D+t-\varepsilon+1}{2}\right)$$

and

$$\begin{aligned}\sum_E \lambda(E,\bar{u})&\leq 2r(\bar{u})-t+2D-3L+\varepsilon-1-s_1-s_2+\sum_{i=1,2}\left(\frac{s_i}{2}-(x_i-1)\right)\\ &\leq 2D-L-4-w+x_1+x_2-t+2D-3L+\varepsilon+1-\frac{s_1}{2}-\frac{s_2}{2}\\ &\leq 2L+\left(4D-6L+\varepsilon-3-w+x_1+x_2-t-\frac{s_1}{2}-\frac{s_2}{2}\right).\end{aligned}$$

We know that $x_i\leq\lfloor(s_i+1)/2\rfloor\leq(2(D-L+t-\varepsilon+1)+1-(L+t-\varepsilon+1))/2\leq(t+2+2D-3L-\varepsilon)/2$. The lemma holds if ($t\geq 2$) or ($\varepsilon=0$) or ($t=1$, $D\equiv 2 \pmod 3$). The only cases left are ($t=0$, $D\equiv 2 \pmod 3$) and ($t=0,1$, $D\equiv 1 \pmod 3$). If $t=0$ and $D\equiv 2 \pmod 3$, we have $x_1=x_2=1$ and $2r(\bar{u})\leq 2D-L-w\leq 2L$. If $t=0,1$ and $D\equiv 1 \pmod 3$ we have $x_i\leq 2$ and

$$2r(\bar{u})\leq 2D-L-4-w+2x_1+2x_2.$$

Since L is even, we have

$$2r(\bar{u})\le 2D-L-6+2x_1+2x_2\le 2L.$$

Therefore, we conclude

$$\sum_E \lambda(E,\bar{u})\le 2L.$$

Case (b.2): $W_2=\emptyset$. Fact 4.3 implies

$$2D-L-\varepsilon+t-1+2x_1\ge 2r(\bar{u})+|P-P(u^*)|-|P(u^*)-P|\ge 2r(\bar{u}).$$

We define

$$k=L-t+\varepsilon-1-|\{E\in P(\bar{u}):\ |Z(E)|=1\}|,$$

$$k'=\frac{2D-L-\varepsilon+t-1}{2}+x_1-r(\bar{u}).$$

It is easy to see $2k'\ge(t+w-\varepsilon+1)-2|P(u^*)-P|$.

$$\begin{aligned}\sum_E \lambda(E,\bar{u})&\le r(\bar{u})+|\{E:\ |Z(E)|=1\}|+\left(\frac{s_1}{2}-(x_1-1)\right)\\&\le\frac{2D-L-\varepsilon+t-1}{2}+x_1-k'+L-t+\varepsilon-1-k+\frac{s_1}{2}-(x_1-1)\\&\le 2L+\left(\frac{2D-3L+\varepsilon-t-1}{2}-k'-k+\frac{s_1}{2}\right).\end{aligned}$$

Therefore we have $\sum_E \lambda(E,\bar{u})\le 2L$ except for the only remaining case that $\varepsilon\ne 0$ and

$$s_1\ge t+2-\varepsilon+2k+2k'+3L-2D. \tag{6}$$

By Fact 4.1, $s_1\le 2D-3L+t-\varepsilon+1$ and we have $0\le k+k'\le\frac{1}{2}$ if $D\equiv 2 \pmod 3$ and $k+k'\le\frac{1}{2}$ if $D\equiv 1 \pmod 3$. We will show by contradiction that this case won't happen. Let w_1, w_2 denote the endpoints of the segment S containing $\bar{u}$ and w_1 is in $P(u)$. We consider the graph formed by deleting one edge $\{w_2,w_2'\}$ in S. If the shortest path P' in $G-\{\{w_2,w_2'\}\}$ from w_1 to $c'=\{a,b\}-P(u)$ does not pass through E, then

$$2D+1\ge 2(L-t+\varepsilon-1-k)+L+t-\varepsilon+1+s_1+1+L-t+\varepsilon.$$

This implies

$$4L-2D-2t+2\varepsilon-1+s_1-2k\le 0.$$

Using (6), we have

$$1\le L-t+\varepsilon\le 4D-6L-1-2k'.$$

If all segments are of length 1, we get $e\ge n+\sum_i(d_i-2)/2\ge 3n/2\ge n-1+(n-1-\varepsilon)/L$ since $L=\lfloor 2D/3\rfloor\ge 3$ for $D\ge 5$. We may assume $L-t+\varepsilon\ge 2$ and $D\equiv 1$ (mod 3). We have

$$2D-3L+t-\varepsilon+1 \geq s_1 \geq t+2-\varepsilon+2k+2k'+3L-2D.$$

Since L is even, we have

$$2D-3L+t-1 \geq t-1+2k+2k'+3L-2D.$$

Therefore

$$2 \leq L-t+\varepsilon \leq 4D-6L-2-2k'.$$

This implies $k'=0$ and $t=L-1$. Now

$$\begin{aligned} r(\bar{u}) &= \frac{2D-L-\varepsilon+t-1}{2}+x_1-k' \\ &= \frac{3L+t}{2}+x_1 \\ &\geq \frac{4L-1}{2}+x_1 \end{aligned}$$

which is impossible.

Suppose P' passes through E. Consider edges in $P(u')-P(\bar{u})$ for $u' \in Z(E)-\{\bar{u}\}$. We have, using Fact 4.1,

$$2D+1 \geq (L+t-\varepsilon+1+s_1+1-|P(w_1)|)+(r(\bar{u})+1-|P(w_1)|)+L-t+\varepsilon.$$

Therefore

$$2\,|P(w_1)| \geq r(\bar{u})+s_1-t+2-2D+2L.$$

Using Fact 4.1 again for bounding the distance from w_2' and some vertex in S^* in $G-\{\{w_2, w_2'\}\}$, we have

$$\begin{aligned} 2D+1 &\geq 2(L-t+\varepsilon-1-k)+2\,|P(w_1)|+t+w-\varepsilon+1 \\ &\quad -2(x_1-1)+L+t-\varepsilon \\ &\geq 2(L-t+\varepsilon-1-k)+2D-L+t-1+2x_1-2k'+s_1-t+2-2D \\ &\quad +2L+t+w-\varepsilon+1-2(x_1-1)+L+t-\varepsilon. \end{aligned}$$

Therefore

$$2k+2k' \geq L+1+s_1+3L-2D \geq 4+s_1+3L-2D \quad \text{since } L \geq 3.$$

$$\begin{aligned} \sum_E \lambda(E,\bar{u}) &\leq r(\bar{u})+L-t+\varepsilon-1-k+\frac{s_1}{2}-(x_1-1) \\ &\leq \frac{2D-L-\varepsilon+t-1+2x_1-2k'}{2}+L-t+\varepsilon-1-k+\frac{s_1}{2}-(x_1-1) \\ &\leq 2L-k-k'-\frac{t}{2}+\frac{\varepsilon}{2}-\frac{1}{2}+\frac{s_1}{2}+\frac{2D-3L}{2} \leq 2L. \end{aligned}$$

This completes the proof of the lemma. □

Combining (4) and Lemmas 5.1-5.3, we obtain immediately the following

$$\begin{aligned}2(e-n+1)L-2(n-1-\varepsilon) &\geq 2xL-2\,|E(F)|+2t-2L\\ &\geq \sum_{\bar{u}}\Big(2L-\sum_{E}\lambda(E,\bar{u})\Big)+2t-2L\\ &\geq 2t-2L.\end{aligned}$$

Therefore

$$\begin{aligned}e &\geq n-1+\frac{n-1-\varepsilon}{L}-\frac{L-t}{L}\\ &\geq n-2+\frac{n-1-\varepsilon}{L}.\end{aligned} \tag{7}$$

6. Fine-tuning

In order to salvage the extra 1 in (7), some further fine-tuning is in order. Although the techniques are quite similar to that of the preceding sections, the details are somewhat complicated. Recall that $\bar{u}_1$ is the leaf-dual on the shortest path S_1 joining a and b in $G-S^*$ and $r(\bar{u}_1)=t+w-\varepsilon$ where $w\geq 1$. For a leaf u, let $S(u)$ denote the segment containing $\bar{u}$.

We consider

$$\begin{aligned}&\sum_{\bar{u}}\Big(2L-\sum_{E\in P(\bar{u})}\lambda(E,\bar{u})\Big)\\ &\quad=\Big(2L-\sum_{E\in P(\bar{u}_1)}\lambda(E,\bar{u}_1)\Big)+\sum_{\bar{u}\neq\bar{u}_1}\Big(2L-\sum_{E}\lambda(E,\bar{u})\Big).\end{aligned}$$

To establish (4), it suffices to show

$$\sum_{\bar{u}\neq\bar{u}_1}\Big(2L-\sum_{E}\lambda(E,\bar{u})\Big)\geq \sum_{E\in P(\bar{u}_1)}\lambda(E,\bar{u}_1)-2t. \tag{8}$$

Since $|r(\bar{u})|=t+w-\varepsilon$, it is enough to show

$$\sum_{\bar{u}\text{ short}}\Big(2L-\sum_{E}\lambda(E,\bar{u})\Big)\geq 2(w-\varepsilon). \tag{9}$$

We may assume $w>\varepsilon$. From Fact 4.8, there is a leaf-dual $\bar{u}$ satisfying $\bar{u}\neq\bar{u}_1$ and $r(\bar{u})=2D-2L+t-w-\varepsilon-q$, where $q\geq 0$.

Suppose no edge is $\bar{u}$-admissible. Then

$$\begin{aligned}2L-\sum_{E}\lambda(E,\bar{u}) &\geq L+t-\varepsilon+1-r(\bar{u})\\ &=L+t-\varepsilon+1-(2D-2L+t-w-\varepsilon-q)\\ &\geq w-\varepsilon+q+(3L-2D+\varepsilon+1).\end{aligned} \tag{10}$$

Fact 6.1. *Suppose $r(\bar{u}) \leq 2D-2L+t-w-\varepsilon$ and some edge is $\bar{u}$-admissible. We have $2L-\sum_E \lambda(E,\bar{u}) \geq w-\varepsilon$ if $L-t+\varepsilon-1 \geq t+w-\varepsilon$. For $L-t+\varepsilon-1<t+w-\varepsilon$, we have $2L-\sum_E \lambda(E,\bar{u}) \geq (L-2t+w-1)/2$.*

Proof. Suppose $r(\bar{u})=2D-2L+t-w-\varepsilon-q$. Suppose E is a $\bar{u}$-admissible edge in $P(\bar{u})$. Let w_1, w_2 denote the endpoints in $S=S(\bar{u})$. We consider the graph formed by removing one edge $\{w_2, w_2'\}$ in S where w_1 and E are in $P(u)=P(u,a)$. Using the same argument as in obtaining (6), the shortest path P' in $G-\{\{w_2,w_2'\}\}$ from w_1 to $c=\{a,b\}-P(u)$ does not pass through E. Therefore

$$2D+1 \geq 2(|S|-1)+\min\{r(\bar{u}'):\ \bar{u}' \in Z(E),\ \bar{u}' \neq \bar{u}\}+1+L-t+\varepsilon.$$

We define $k=L-t+\varepsilon-1-(|S|-1)$.

$$\begin{aligned}2D+1 &\geq 2(L-t+\varepsilon-1)-2k+L+t+s-\varepsilon+1+1+L-t+\varepsilon\\ &\geq 4L-2t+2\varepsilon-2k+s.\end{aligned}$$

This implies

$$2k \geq 4L-2D-2t+2\varepsilon-1+s.$$

Suppose $P(u^*)$ does not contain special edges. We have

$$\begin{aligned}\sum_E \lambda(E,\bar{u}) &\leq r(\bar{u})-|S|-1+\frac{s}{2}\\ &\leq 2D-2L+t-w+\varepsilon-q+L-t+\varepsilon-1-k+\frac{s}{2}\\ &\leq 2L-w+\varepsilon-k+\frac{s}{2}-q+2D-3L-\varepsilon-1\\ &\leq 2L-(w-\varepsilon)-\frac{4L-2D-2t+2\varepsilon-1}{2}-q+2D-3L-\varepsilon-1\\ &\leq 2L-\frac{(L-2t+w-1)}{2}-q.\end{aligned}$$

Therefore

$$2L-\sum_E \lambda(E,\bar{u}) \geq w-\varepsilon+q \quad \text{if } L-t+\varepsilon-1 \geq t+w-\varepsilon.$$

Suppose both $P(u)$ and $P(u^*)$ contain $\bar{u}$-special edges. We follow the notation in the proof of Lemma 5.3 (for the definitions of W_i, x_i, r_i and s_i, $x_1 \leq x_2$ (if $x_1=x_2$, we choose $s_1 \geq s_2$)). Suppose (5) holds. We have $r(\bar{u}) \leq 2D-2L-2+2x_1-s_1$ and, using Fact 4.5,

$$\begin{aligned}k &\geq L-t+\varepsilon-1-(2D-2L-2+2x_1-2s_1-s_2-t+\varepsilon-1-3L+2D)\\ &\geq 2s_1+s_2-2x_1-4D+6L+2,\end{aligned} \tag{11}$$

$$\sum_E \lambda(E,\bar{u}) \le r(\bar{u}) + L - t + \varepsilon - 1 - k + \frac{s_1}{2} - (x_1 - 1) + \frac{s_2}{2} - (x_2 - 1)$$

$$\le 2D - 2L + t - w - \varepsilon - q + L - t + \varepsilon - 1 - \frac{3}{2}s_1 - \frac{s_2}{2} + 4D - 6L.$$

We have $2L - \sum_E \lambda(E,\bar{u}) \ge (w-\varepsilon) + q$ except for $D \equiv 1 \pmod 3$ and $3s_1/2 + s_2/2 < 4$. It remains to check two cases ($s_1 = 2$, $s_2 = 1$) and ($s_1 = s_2 = 1$). Both can be easily verified.

Suppose (5) does not hold. We have, as in Case (b) of Lemma 5.3,

$$2D - L - 4 - w + 2x_1 + 2x_2 \ge 2r(\bar{u}).$$

From Fact 4.5, we have

$$|\{E \in P(\bar{u}) \colon |Z(E)| = 1\}| \le r(\bar{u}) - t - s_1 - s_2 - 3L + 2D - 1$$

and

$$\sum_E \lambda(E,\bar{u}) \le 2r(\bar{u}) - t - s_1 - s_2 - 3L + 2D - 1 + \frac{s_1}{2} - (x_1 - 1) + \frac{s_2}{2} - (x_2 - 1)$$

$$\le 2L - (w-\varepsilon) + x_1 + x_2 - 3 - 6L + 4D - t - \frac{s_1}{2} - \frac{s_2}{2} - \varepsilon.$$

Therefore $\sum_E \lambda(E,\bar{u}) \le 2L - (w-\varepsilon)$. (For $\varepsilon = 1$ and $t = 0$, by Fact 4.5, we have

$$|\{E \in P(\bar{u}) \colon |Z(E)| = 1\}| < r(\bar{u}) - t - s_1 - s_2 - 3L - 2D - 2.)$$

For $L - t + \varepsilon - 1 < t + w - \varepsilon$, the above inequality implies

$$\sum_E \lambda(E,\bar{u}) \ge 2L - \frac{L - 2t + w - 1}{2}.$$

This completes the proof of the fact. □

A leaf-dual $\bar{u}$ is said to be good if $r(\bar{u}) \le 2D - 2L + t - w - \varepsilon$. We are now ready to complete the proof for the main theorem by establishing (8). There are two cases.

Case 1: $L - t + \varepsilon - 1 \ge t + w - \varepsilon$ *or* S_1 *contains no* $\bar{u}_1$*-admissible edges.* For $L - t + \varepsilon - 1 \ge t + w - \varepsilon$, it suffices to establish (9).

If S_1 contains no admissible edge,

$$\sum_{E \in P(\bar{u}_1)} \lambda(E,\bar{u}_1) \le r(\bar{u}_1) + L - t + \varepsilon - 1$$

$$\le L + w - 1.$$

To establish (8) it is enough to show

$$\sum_{\bar{u} \ne \bar{u}_1} \left(2L - \sum_E \lambda(E,\bar{u})\right) \ge L - 2t + w - 1, \tag{12}$$

if $L - t + \varepsilon - 1 < t + w - \varepsilon$ and S_1 contains no admissible edge.

If there are at least two good leaf-duals, (9) or (12) holds using Fact 6.1. We may assume there is only one good leaf-dual $\bar{u}$ with $r(\bar{u})=2D-2L+t-w-\varepsilon-q$, $q\geq 0$. It suffices to show that there exist leaf-duals $\bar{u}$ and $\bar{u}'\neq\bar{u}_1$ such that

$$4L-\sum_E \lambda(E,\bar{u})-\sum_E \lambda(E,\bar{u}')\geq L-2t+w-1, \tag{13}$$

which implies (12) and also implies (9) for $L-t+\varepsilon-1\geq t+w-\varepsilon$.

If there is only one good leaf-dual, a and b must both have degree 3 by using Fact 4.8. Let w_3 be the vertex on $P(u)$ of degree at least 3 closest to a. Using the same argument as that of Fact 4.8, $d(w_3,a)=D-L-w-\lceil(q+1)/2\rceil+l$ for $l\geq 0$ since there is only one good leaf-dual. Without loss of generality, we may assume the distance of a degree ≥ 3 vertex w_4 (if any) from a or b on $P(\bar{u})$ is no smaller than $d(w_3,a)$. $Z(w_3)$ contains a leaf u', $u'\neq u$ and we consider the distance from a' (which is adjacent to a in S^*) to vertices in $P(\bar{u}')-P(a,w_3)$, after deleting the edge $\{a,a'\}$. We have

$$r(\bar{u}')+1-d(a,w_3)+(r(\bar{u})+1-d(a,w_3))\leq 2D-2L+2t-2\varepsilon+3.$$

We get

$$\begin{aligned}r(\bar{u}')&\leq 2D-2L+2t-2\varepsilon+1-(2D-2L+t-w-\varepsilon-q)+2(D-L-w)\\&\quad+2l-2\left\lceil\frac{q+1}{2}\right\rceil\\&\leq L+t+q+2l-w+2D-3L-\varepsilon+1-2\left\lceil\frac{q+1}{2}\right\rceil.\end{aligned}$$

Therefore by Fact 6.1 and (10) if $L-t+\varepsilon-1\geq t+w-\varepsilon$, $2L-\sum_E \lambda(E,\bar{u}')\geq w-2l-q+3L-2D+2\lceil(q+1)/2\rceil$. Let k denote $L-t+\varepsilon-1-|\{E\in P(\bar{u}): Z(E)=1\}|$. We have

$$\begin{aligned}k&\geq L-t+\varepsilon-1-(r(\bar{u})-2d(a,w_1))\\&\geq L-t+\varepsilon-1-(2D-2L+t-w-\varepsilon-q)+2(D-L-w+l)+\left\lceil\frac{q+1}{2}\right\rceil\\&\geq(L-2t-w)+2\varepsilon-1+q+2l+\left\lceil\frac{q+1}{2}\right\rceil.\end{aligned}$$

If $P(a,w_3)$ contains no $\bar{u}$-admissible edges, we have

$$\begin{aligned}2L-\sum_E \lambda(E,\bar{u})&\geq 2L-(2D-2L+t-w-\varepsilon-q)-(L-t+\varepsilon-1)\\&\quad+(L-2t-w)+2\varepsilon-1+2l+q+2\left\lceil\frac{q+1}{2}\right\rceil\\&\geq L-2t+2\varepsilon+q+2l+3L-2D+2\left\lceil\frac{q+1}{2}\right\rceil.\end{aligned}$$

Therefore

$$4L-\sum_E \lambda(E,\bar{u})-\sum_E \lambda(E,\bar{u}')\geq L-2t+w-1$$

and (13) holds.

Suppose $P(a, w_3)$ contains a $\bar{u}'$-admissible edge E. Then $s_1 = r(\bar{u}) - (L+t-\varepsilon+1) = 2D-3L+2l+q-w-1$. If $P(u^*, b)$ does not contain admissible edges, then

$$\begin{aligned}
2L - \sum_E \lambda(E, \bar{u}) &\geq (w-\varepsilon) + k - \frac{s_1}{2} \\
&\geq w - \varepsilon + (L-2t-w) + 2\varepsilon - 1 + q + 2l + 2\left\lceil \frac{q+1}{2} \right\rceil \\
&\quad - \frac{2l+q-1-w+2D-3L}{2} \\
&\geq (w-\varepsilon) + (L-2t-w) + \frac{w}{2} + \frac{2l+q+1}{2} + 2\varepsilon - \frac{2D-2L}{2} \\
&\geq 2(w-\varepsilon) + (L-2t-w+2\varepsilon-1).
\end{aligned}$$

If $P(u^*, b)$ contains admissible edges, then

$$\begin{aligned}
2L - \sum_E \lambda(E, \bar{u}) &\geq w - \varepsilon + k - \frac{s_1}{2} - \frac{s_2}{2} \\
&\geq (w-\varepsilon) + (L-2t-w) + 2\varepsilon - 1 + q + 2l + 2\left\lceil \frac{q+1}{2} \right\rceil \\
&\quad - (2l+q-1-w+2D-3L) \\
&= 2(w-\varepsilon) + (L-2t-w+2\varepsilon-1).
\end{aligned}$$

Case 1 is proved.

Case 2: $L-t+\varepsilon-1 < t+w-\varepsilon$ *and* S_1 *contains special edges*. Let E' denote a special edge in S_1 with the least rank. Without loss of generality we assume E' is in $P(u_1)$. Suppose u'' is in $Z(E') - \{u_1\}$ with $r(u'') = L+t-\varepsilon+1+s''$. Let S' denote the segment containing u''. Using the same argument as in Fact 4.8, we have $k_1' = L-t+\varepsilon-1-(|S'|-1)$ satisfying $2k_1' \geq L-2t+\varepsilon-1+s''$. Suppose $P(u_1^*)$ does not contain special edges. We have

$$\begin{aligned}
\sum_E \lambda(E, \bar{u}_1) &\leq t + w - \varepsilon + |S'| - 1 + \frac{s''}{2} \\
&\leq t + w - \varepsilon + L - t - \varepsilon - 1 - k_1' + \frac{s''}{2} \\
&\leq L + w - 1 - \frac{L-2t+\varepsilon-1}{2} \\
&\leq \frac{L+2t-\varepsilon-1}{2} + w.
\end{aligned}$$

Fact 4.8 guarantees the existence of at least one good leaf-dual, by Fact 6.1 we have

$$\sum_{u\neq\bar{u}}\left(2L-\sum_{E}\lambda(E,\bar{u})\right)\geq(w-\varepsilon)+\frac{L-2t+\varepsilon-1}{2}$$
$$\geq\sum_{E}\lambda(E,\bar{u}_1)-2t.$$

Therefore (8) holds.

Suppose $P(u_1^*)$ also contains special edges. This implies

$$\sum_{E}\lambda(E,\bar{u}_1)\leq 2(t+w-\varepsilon)-t-s_1-s_2+2\varepsilon-1+\frac{s_1}{2}+\frac{s_2}{2}$$
$$\leq t+2w-\frac{s_1+s_2}{2}-1.$$

It suffices to show

$$(w-\varepsilon)+\frac{L-2t+\varepsilon-1}{2}\geq 2w-t-\frac{s_1+s_2}{2}-1$$

which holds because of Fact 4.7. This completes the proof for the main theorem.

7. The general case

In this section, we briefly discuss the general case when more than one edge is to be removed from G. Suppose after deleting s edges, the remaining graph still has diameter D. How many edges must G have?

First of all, G must be $(s+1)$-edge-connected. This implies every vertex has degree $\geq s+1$. Therefore G has at least $(s+1)n/2$ edges.

A result of Bollobás and de la Vega [7] states that a random $(s+1)$-regular graph has diameter at most

$$\log_s n+\log_s\log n+c$$

where c is some small constant <10.

Using the proof in [7], one can show there is an $(s+1)$-regular graph F on $X=s^{d-1}/d$ vertices having diameter d and by deleting any s edges the remaining graph has diameter $\leq d+2$ for large d.

We can now form our graph G by combining copies of such graphs on $cs^{D/2-3}/D$ vertices by identifying one vertex from each copy. The resulting graph G has the property that after deleting any s edges the diameter is still $\leq D$ and G has $(s+1)n/2+c(s+1)nD/s^{D/2-3}$ edges. Therefore

$$g(n,D,s)\leq(s+1)n/2+cn/s^{D/2-4}\quad\text{for large } n.$$

We note that slightly better examples can be easily obtained by sharpening the estimate of the result in [7]. We also remark that instead of using random graphs as F we can use graphs constructed explicitly (such as expander graphs) together with a random matching as discussed in [6]. These graphs also have near optimum diameters (even after the deletion of a small number of edges).

Concluding remarks

A natural question is to determine $g(n, D, s)$, the minimum number of edges in a graph G with the property that after deleting any s edges the remaining graph has diameter D. In particular, it is of interest to determine $g(n, D, 2)$, which is considerably harder than the case of $s = 1$.

Another direction is to consider further constraints on the graphs, such as requiring the degree to be small. What is the minimum number of edges in a graph on n vertices with maximum degree k with the property that after deleting s edges the remaining graph still has diameter D? We can let $g_k(n, D, s)$ denote such minimum value. It would be of interest to determine $g_k(n, D, 1)$.

There is also an analogous version of vertex deletion. Let $f(n, D, s)$ denote the minimum number of edges in a graph with the property that after deleting any s vertices the remaining graph has diameter D. Clearly $g(n, D, s) \geq f(n, D, s)$. Various partial results on $f(n, D, s)$ can be found in [5]. The problem of determining $f(n, D, s)$ seems to be as hard if not harder than the problem of determining $g(n, D, s)$. For $s = 1$, the problem of determining $f(n, D, 1)$ still remains open. The value of $g(n, D, s)$ that is determined in this paper provides the best known upper bound for $f(n, D, 1)$ so far.

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