DIAMETERS OF GRAPHS: OLD PROBLEMS AND NEW RESULTS

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ABSTRACT

We will discuss several interrelated problems on diameters of graphs (i.e., the maximum distance among all pairs of vertices). Of particular interest are graphs with small diameters after deleting few edges. We will investigate extremal and algorithmic aspects of these problems as well as their applications in communication networks.

I. Introduction

In a graph G with vertex set $V(G)$ and edge set $E(G)$, the *distance* $d(u,v)$ between two vertices u and v is the length of a shortest path joining u and v. The *diameter* of G, denoted by $D(G)$, is the maximum distance among all pairs of vertices in G.

Diameter problems arise in network optimization in a natural way. As early as the 1960's, Erdős and Rényi [32] asked the problem of scheduling airplane flights between n cities so that it is possible to fly from any one city to another with only a few intermediate stopovers along the way (subject to capacity constraints of the airports). Many problems in data communication or circuit layout optimization can often be related to a few key invariants of which the diameter is an important one. In particular, when the number of edges in a path is roughly proportional to the delay or signal degradation encountered by messages sent along the path, the diameter is proportional to the worst case performance bounds.

Here we will give several similar but somewhat different versions of the diameter problems. The first two problems were attributed to Elspas in a 1964 paper [31]. How-

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ever, they can be traced back to E. F. Moore in 1960 (see [41]):

Problem 1: Given k and D, construct a graph with as many vertices as possible with maximum degree k and diameter D.

Problem 2: Given n and k, construct a graph with minimum diameter on n vertices and maximum degree k.

These two versions offer a somewhat different point of view when evaluating various constructions (detailed later in Section 2).

Vijayan and Murty [55] investigated the reliability of graphs with diameter constraints. They posed the following problem:

Problem 3: Given n, D, D' and s, what is the minimum number of edges in a graph on n vertices of diameter D with the property that after removing s edges the remaining graph has diameter no more than D'?

Of course, there is an analogous version with degree constraints.

Problem 4: Given n, D, D', k and s, what is the minimum number of edges in a graph on n vertices with maximum degree k, diameter D, satisfying the property that after removing s edges the remaining graph has diameter no more than D'?

These problems have attracted the attention of many researchers. There is a large literature described in survey papers [2,3,5,24,27,44]. However, the solutions are far from satisfactory and most of the problems remain unsolved.

Instead of edge deletion, we can also consider the complementary problem of edge augumentation.

Problem 5: For a given graph G, it is of interest to find the optimum way to add t edges so that the resulting graph has minimum diameter.

Problem 6: For a given graph, try to add t vertex disjoint edges to reduce the diameter as much as possible.

In this paper, we briefly survey the known results, report on recent progress in the area and point out some interesting open problems. This paper is organized into seven sections. In Section 2, we discuss explicit constructions, probabilistic methods and a third "hybrid" approach. In Section 3, diameter bounds for various altered graphs are examined. In Section 4, we study edge-minimum graphs with small diameters after edge deletion. In Section 5, we investigate the extremal graphs of Problem 3 and 4. In particular, we focus on methods for constructing graphs with small diameter and degree. In Section 6, we consider diameter algorithms and computational complexity. In Section 7, we conclude with several generalizations and variations of diameters.

2. Pseudo-random constructions

For many combinatorial problems, the following intriguing situation occurs: It is very difficult to construct explicitly a "good" configuration. However, by probabilistic arguments, it can be shown that "almost all" configurations are good. By an explicit construction, we mean a deterministic method which produces arbitrarily large graphs, and which does not depend on any random process. One notorious example is that of Ramsey graphs. One can easily show that if the edges of a complete graph on n vertices are randomly colored red or blue, then the largest red clique or blue clique is of size $c \log n$. However, in all known constructions, monochromatic cliques of size $e^{c\sqrt{\log n}}$ cannot be prevented (see [34]).

Why are explicit constructions desirable? On one hand, random constructions can be easier to analyze probabilistically. However, random graphs are harder to control, in the sense that one needs n^2 entries to describe the edges, whereas a systematic approach may use much less memory (as in a recent paper [28] which requires only k integers to construct a k-regular "good" graph). In addition, after choosing a random graph, an efficient testing scheme is required to ensure that the graph is indeed "good". Furthermore, in many problems, explicit constructions are much better for other reasons such as actually finding paths from one vertex to another as required in many routing and sorting problems.

For the diameter problems that we are interested in here, it turns out that random graphs often provide near optimum solutions. Recall the first problem of minimizing the diameter for a given number of vertices n and degree k. It is easy to establish a lower bound, the so-called Moore bound, for the diameter. Namely, for any vertex v there are at most k vertices at distance 1 and, in general, there are at most $k(k-1)^{i-1}$ vertices at distance i.

Therefore,

$$n \leq 1 + k + \cdots + k(k-1)^{i-1} + \cdots + k(k-1)^{D-1}$$

and we get

$$D \geq \log_{k-1} n - \frac{2}{k}$$

On the other hand, Bollobás and de la Vega [16] proved the following

Theorem 2.1. [16] A random k-regular graph has diameter

$$\log_{k-1} n + \log_{k-1} \log n + c$$

with probability close to 1, where c is a small constant (at most 10) and all k-regular graphs are considered to have equal probability.

So, a random regular graph usually has its diameter very close to the optimum. To look at this from the view of Problem 2, it follows that a random regular graph has degree k and diameter D if the number of vertices is at least $(k-1)^{D-1}/(2k\log(k-1))$ which is fairly close to the Moore bound.

The best explicit construction for graphs with degree k and diameter D are the de Bruijn graphs and their variations which have very simple and nice structure. The basic construction consists of the vertex set $\{(a_1, \ldots, a_r) : a_i = 1, 2, \ldots, s\}$, with

$(a_1, \ldots, a_r)$ adjacent to $(a_2, \ldots, a_r, b)$ for any b. Such graphs have degree $k = 2s$ and diameter $D = r$ with $n = \left\lfloor \frac{k}{2} \right\rceil^D$ vertices. The number of vertices is off by a factor of 2^D from the Moore bound (which may be closer to the right answer). For small values of D and k, better constructions exist which will be discussed further in Section 5.

In the past few years, much progress has been made on constructive methods for another important combinatorial problem -- the construction of expander graphs. Expander graphs come up in a variety of contexts such as sorting, permutation networks, computational complexity, and many extremal graph problems. Roughly speaking, an expander graph is a graph G with the property that for any subset S of vertices with $|S|$ not too large, say $|S| < n/2$, the number of neighbors $N(S)$ of S is large, say $N(S) \geq c\,|S|$ where $c > 1$ depends only on the maximum degree of G.

Many interesting techniques have been used to prove the expanding properties of various constructions; see Margulis [47], Gabber and Galil [38], Tanner [53] and Alon [1]. In particular, Tanner [53] pointed out the following relationship between the second largest eigenvalue and the expanding property of a k-regular graph. For any set X of vertices, the number of neighbors $N(X)$ of X satisfies

$$N(X) \geq \frac{k^2 |X|}{(k^2 - \lambda^2)\,|X|/n + \lambda^2} \tag{1}$$

Recently, Lubotzky, Phillips and Sarnak [46] have constructed k-regular graphs with λ satisfying $|\lambda| \leq 2\sqrt{k-1}$, which is best possible. These graphs, which they term Ramanujan graphs, have diameter $2\log_{k-1} n + c$, which is about twice as large as the optimum. Although these graphs compare unfavorably with de Bruijn graphs (as far as diameters are concerned), the expanding properties lead to a third approach.

The pseudo-random approach can be viewed as a "half-way" solution by blending a good construction with a small amount of randomness. Recently, Bollobás and Chung [17] proved that the graph obtained by adding a random matching to the n-cycle C_n has diameter very close to the optimum value. (A matching is a maximum set of vertex-disjoint edges). They also prove a general theorem which asserts that by adding a random matching to a $(k-1)$-regular graph with certain expanding properties (for example if $\lambda < k - \epsilon$ for any positive ϵ, the expanding property by (2) is enough), the resulting graph has diameter about $\log_{k-1} n$, which is the correct order of the best possible value. In particular, therefore, adding a random matching to the Ramanujan graph results in a graph with near optimum diameter. Further discussions on the change of diameters while adding edges (not necessarily a matching) or deleting edges will be given in the next section.

3. Diameter bounds for altered graphs

We now present a sequence of examples starting with the path P_n on n vertices. How small can the diameter be made by adding t edges to the path P_n? Clearly the diameter of P_n is $n-1$ and, by adding one edge to P_n, we can form the cycle C_n which is of diameter $\left\lfloor \frac{n}{2} \right\rfloor$. What if we add two edges to P_n? Suppose we first add one edge to form C_n and then add the second edge (in any way), the resulting graph has diameter $\left\lfloor \frac{n}{2} \right\rfloor$. However, the minimum diameter by adding two edges to P_n is approximately $\frac{n}{3}$, which is much smaller than $\left\lfloor \frac{n}{2} \right\rfloor$ for large n. The following results of Chung and Garey [24] state:

Theorem 3.1. [24] By adding t edges to P_n the resulting graph always has diameter D satisfying $D \geq \frac{n-t}{t+1}$. Also, there is a way of adding t edges so that $D \leq \frac{n}{t+1} + 3$.

Theorem 3.2. [24] By adding t edges to a graph of diameter $D(G)$, the resulting graph always has diameter D' satisfying

$$D' \geq \frac{D(G)-t}{t+1}$$

Furthermore, the extremal cases (in this case, the minimum diameter over all graphs with diameter D plus t edges) are achieved by paths.

This makes it more interesting to study the minimum diameter $D(P_n, t)$ obtained by adding t edges to the path P_n. Recently, Schoone, Bodlaender and van Leeuwen [52] improved these bounds for $D(P_n, t)$ and very recently, Kerjouan [43] further improved the bounds to

$$\frac{n+t-1}{t+1} \geq D(P_n, t) \geq \frac{n+t-4}{t+1} \quad \text{for } t \text{ even },$$

and

$$\frac{n+2t-5}{t} \geq D(P_n, t) \geq \frac{n+t-4}{t+1} \quad \text{for } t \text{ odd }.$$

Still, the exact values are not yet determined. It seems reasonable to conjecture the following:

Conjecture 3.3

$$f(P_n, t) = \frac{n+t-1}{t+1} \quad \text{for } t \text{ even },$$

and

$$f(P_n, t) = \frac{n+2t-5}{t+1} \quad \text{for } t \text{ odd }.$$

The problem of adding edges to an n-cycle C_n is also interesting. Sometimes a cycle is preferable for network topologies because of its symmetry and balance. In [24] it was shown that the minimum diameter by adding t edges to C_n is approximately $\frac{n}{t+2}$ when t is even, and $\frac{n}{t+1}$ when t is odd. The best choice of t new edges to add to C_n

to minimize the diameter depends on the parity of t: if t is odd, form a star of t edges; if t is even, form the union of two stars.

Suppose it is required that the resulting graph should have a small maximum degree or be almost regular (for example, the maximum degree differs from the minimum degree by at most 1). Then the resulting graph cannot have diameter $\leq \frac{n}{t}$, as was shown in [17].

Theorem 3.4. [17] By adding a matching (that is, $\lfloor \frac{n}{2} \rfloor$ vertex-disjoint edges), the resulting graph can have diameter at most $\log_2 n + \log\log n + c$ where c is a small constant.

This is very close to the minimum possible diameter for a graph on n vertices and maximum degree 3.

Theorem 3.5. [17] By adding a matching to a $(k-1)$-regular expander graph (or just a $(k-1)$-regular graph with the property that $N_i(X) = \{y : d(X, y) \leq i\}$ has at least $c(k-2)^i$ vertices for some constant c and for any $i \leq \left(\frac{1}{2} + \epsilon\right) \log_{k-1} n$), then the resulting graph can have diameter at most $\log_{k-1} n + \log_{k-1} \log n + c'$ for some constant c'.

Using similar arguments as in [17], it can be easily extended to the following more general case.

Theorem 3.6. By adding t vertex-disjoint edges to P_n or C_n, the resulting graph can have diameter $C \frac{n}{t} \log t$ and this is best possible (to within a constant factor).

4. Extremal problems for altered graphs

In this section, we focus on the extremal problems, namely, Problems 3 and 4. The goal is to minimize the number of edges in a graph so that after deleting any choice of a small but fixed number of edges the resulting graph has small diameter. The original formulation of Vijayan and Murty is the problem of determining $g(n, D, D', s)$, which is the least number of edges in a graph on n vertices with diameter at most D such that after removing any s edges, the remaining graph still has diameter at most D'. From Theorem 3.1 we know that when D' is large, say $D' \geq (s+1)(D+1)$, the value of $g(n, D, D', s)$ is the same as $g(n, D, n-1, s)$. In other words, the problem is reduced to finding minimum-edge graphs on n vertices with diameter D and edge connectivity s. This case was solved by Bollobás [10,14].

Theorem 4.1.

$$g(n, D, D', s) = g(n, D, n-1, s)$$

$$\text{for} \qquad D' \geq (s+1)(D+1)$$

Theorem 4.2. [10]

$$\frac{n}{2}\left(s+1+\frac{s-1}{s^m-1}\right) - c \;<\; g(n, 2m, n-1, s) \;<\; \frac{n}{2}\left(s+1+\frac{s-1}{s^m-1}\right)$$

where c depends on m and s.

The general problem of finding $f(n, D, D', s)$ (especially when D' is not too large in comparison with D) remains far from resolved. There are many papers on this problem, such as Bollobás, [9,10,15], Bondy and Murty [18], Caccetta [20,21,22] and Bollobás and Erdös [13], Bollobás and Eldridge [12] and Bollobás and Harary [14]. Most of the results concern the case $D, D' \leq 4$ and $s = 1$. In the study of $g(n, D, D', s)$, one simplified version is of particular interest.

Problem 4.2: What is the minimum number $g(n, D, s)$ of edges in a graph on n vertices with the property that after removing s edges the remaining graph has diameter at most D?

Even the case of $s=1$ (which was conjectured in [14,15]) was unresolved until very recently.

Theorem 4.3. [27]

$$g(n,D,1) = n-1 + \left\lceil \frac{n-1-\epsilon}{\lfloor \frac{2}{3} D \rfloor} \right\rceil$$

where $\epsilon = 1$ if $D \equiv 1 \pmod 3$, and 0 otherwise.

For general s, it is easy to see that $g(n,D,s) \geq (s+1)n/2$. In [27] it was also shown that there exist graphs which give the following upper bound for $g(n,D,s)$.

Theorem 4.4. [27]

$$g(n,D,s) \leq (s+1)n/2 + c(s+1)n\,D\,s^{-D/2+s+3}$$

for some constant c.

Problem 4.5: Determine $g(n,D,s)$ for $s \geq 2$.

The solutions for $g(n,D,1)$ and for the bound on $g(n,D,s)$, $s \geq 2$, are graphs with some vertices of large degree. It is of interest to ask the analogous questions with degree constraints.

Problem 4.6: What is the least number $g_k(n,D,s)$ of edges in a graph on n vertices and with maximum degree k satisfying the property that after removing any s edges, the remaining graph has diameter at most D?

Problem 4.7: What is the least number $g_k(n,D,D',s)$ of edges in a graph on n vertices of diameter at most D with maximum degree k satisfying the property that after removing any s edges the remaining graph has diameter at most D'?

5. Extremal graphs with bounded degree and small diameter

In this section, we will examine the extremal problems on graphs with as many vertices as possible, say $n(k,D)$ vertices, satisfying degree at most k and diameter at most D. As discussed in Section 2, the maximum number $n(k,D)$ of vertices in a graph with diameter D and maximum degree k can be bounded above by

$$n(k,D) \leq 1+k+\cdots+k(k-1)^{D-1} = n_0(k,D) \quad .$$

The upper bound $n_0(k,D)$, called the Moore bound, is provably unreachable [41] for almost all nontrivial values of k and D. The only graphs, called Moore graphs, which achieve the Moore bounds must be one of the following [41]:

(i) $D=1$, $(k+1)$-cliques

(ii) $k=2$, $(2D+1)$-cycles

(iii) $D=2$ and $k=3$, the Petersen graph

(iv) $D=2$ and $k=7$, the Hoffman-Singleton graph

(v) (possibly) $D=2$ and $k=57$.

Problem 5.1 [41]: Is there a Moore graph of diameter 2 and degree 57?

The current known bounds for $n(k,D)$ for general k and D are indeed very poor. For the upper bound, the only result beyond $n(k,D) < n_0(k,D)$ (except for (i)-(v)) was obtained by P. Erdős, S. Fajtlowicz and A. J. Hoffman [34] who proved

$$n(2k,2) \leq n_0(2k,2)-2 \text{ for } k>1 \quad .$$

Problem 5.2. [34]: Is it true that for every integer c there exist k and D such that $n(k,D) \leq n_0(k,D)-c$?

As for the lower bound, random regular graphs have near optimum diameters [16]. Namely, the probabilistic lower bound for $n(k,D)$ is

$$n(k,D) \geq \frac{1-\epsilon}{2\,k\,D\log(k-1)} \quad (k-1)^{D-1}$$

This bound is considerably better than the constructive lower bound. In the remainder of this section we focus on the constructive lower bounds.

For larger k and D, the constructions are mainly de Bruijn graphs and their variations. We recall that the de Bruijn graph $B(r,s)$, for given integers r and s, has s^r vertices represented by r-tuples $(a_1, a_2, \cdots, a_r)$ where $a_i \in \{1, \cdots, s\}$, and $(a_1, a_2, \ldots, a_r)$ is adjacent to $(a_2, \cdots, a_r, b)$ and $(b, a_1, \ldots, a_{r-1})$ for any $b \in \{1, \ldots, s\}$. It is easy to see that $B(r, s)$ has diameter r and degree $2s$. This gives, constructively,

$$\mu_D = \limsup_{n \to \infty} N(k, D)/N_0(k, D) \geq 2^{-D}$$

(We note that in considering constructive lower bounds, it is sometimes only required to establish an infinite set of good constructions instead of requiring good constructions for each n. Therefore we consider the *lim sup* of $n(k, D)/n_0(k, D)$ instead of *lim inf* as in some other papers [2,3]).

One of the major problems on this topic is to improve the constructive lower bounds.

Problem 5.8: **For each D, construct graphs for infinitely many values of k with $(2-\epsilon)^{-D}\, n_0(k, D)$ vertices and of degree at most k, diameter at most D for some fixed $\epsilon > 0$.**

For some small fixed values of D, there are explicit constructions of large classes of graphs with roughly $n_0(k, D)$ vertices (asymptotically for large k) using combinatorial structures known as generalized $n-gons$, and various product constructions [4,6]. The best known constructive lower bounds μ_D, for $D \leq 10$, are listed here.

D	1	2	3	4	5	6	7	8	9	10
n_D	1	1	1	$3^3\, 2^{-7}$	1	$2 \cdot 5^5\, 6^{-6}$	1	$2 \cdot 7^7\, 8^{-8}$	$4 \cdot 7^7\, 9^{-9}$	$5 \cdot 2^{-10}$

The de Bruijn graph can be improved slightly by taking the following induced subgraph $B'(r,s)$ (also known as the Kautz graph). The vertices of $B'(r,s)$ are r-tuples $(a_1, a_2, \ldots, a_r)$ with $a_i \neq a_{i+1}$ for $1 \leq i \leq r-1$ and $a_1 \neq a_r$. It is easy to see that

$B'(r,s)$ has $s(s-1)^{t-1}$ vertices and of degree $2(s-1)$, which compares favorably to $B(r,s)$. Still, it does not affect the major term asymptotically.

Another technique for improving the constructions is the method of covering codes which can be best illustrated by the following example for the case of $k=3$.

The de Bruijn graph for $k=3$ and general D has a vertex set consisting of all binary r-tuples. A vertex $(a_1,\ldots,a_r)$, $a_i \in \{0,1\}$, is adjacent to $(a_2,\ldots,a_r,a_1)$ to $(a_1,\ldots,a_{r-1},1-a_r)$. Such a graph, denoted by $B_2(r)$, obviously has diameter $2r$ or $2\log_2 n$, where n is the number of vertices. Leland and Solomon [45] gave a complicated proof for a construction with diameter $1.5\log_2 n$. Here we will show a simple construction using covering codes.

A subset S of binary t-tuples $\{0,1\}^t$ is called a covering code of radius R if for any t-tuple v, there exists an element s in S such that the Hamming distance between v and s is no more than R.

We will fold the graph $B_2(t)$ by S as follows: The new graph, denoted by $B_2(t)/S$ has vertices each of which is a set $\{(a_1,\ldots,a_t)+s : s \in S\}$ (The addition is performed componentwise modulo 2). Since such folding preserves edges, the resulting graph has degree at most 3. Furthermore, the graph has $2^t/|S|$ vertices and diameter $t+R$.

If we take S to be two points $(0,0,\cdots,0)$ and $(1,1,\ldots,1)$, it is clear that $R=t/2$. Therefore the diameter is $t+R=1.5t=1.5(\log_2 n+1)$. The idea of folding graphs was suggested by Tom Leighton (private communication).

Jerrum and Skyum [42] constructed graphs on n vertices with degree 3 and diameter $1.47\log_2 n$ by a complex scheme. Here again we can obtain this bound by folding a modified version of a de Bruijn graph.

We start with a graph with a vertex set consisting of all ternary t-tuples and $(t-1)$-tuples. $(a_1,\ldots,a_t)$, $a_i \in \{0,1,2\}$, is adjacent to $(a_2,\ldots,a_t,a_1)$ and to $(a_1,\ldots,a_{t-1})$.

Such a graph $B_3(t)$ has degree 3 with $3^t + 3^{t-1}$ vertices of diameter $3t$. Now we take a covering code S consisting of $(0,0,...,0)$, $(1,1,...,1)$ and $(2,2,...,2)$. Clearly it has radius $2t/3$. After folding $B_3(t)$ by S the resulting graph has a diameter $\frac{7t}{3} = \frac{7}{3}\log_3 n \approx 1.47 \log_2 n$. For $k=3$, $1.47 \log_2 n$ is the best known constructive lower bound.

Problem 5.4: Construct graphs on n vertices with degree 3 and diameter $< (\frac{7}{3}\frac{\log 2}{\log 3} - \epsilon)\log_2 n$ for $\epsilon > 0$.

In general, the idea of folding by covering codes can be extended to a more general setting by considering cosets of isomorphisms on special graphs. Very recently, Mike Fellows and others [36] succeeded in using the Golay code and the folding techniques to construct large graphs for other values of k and D. The following table for the best constructions of graphs with degree at most k and diameter at most D currently known has been improved in many entries compared to those given in the table in [4,6].

6. The computational complexity of determining the diameter

For a given graph G on n vertices a straightforward way to determine the diameter $D(G)$ is as follows:

(1) Find the breadth-first search tree [54] for each vertex v of G. Thereby determine the maximum distance $d_v = \max d(u,v)$.

(2) Compare d_v and determine $D(G) = \max d_v$.

Since the time requirement [54] for finding a breadth-first search tree is $O(n+e)$, the preceding algorithm has complexity $O(n^2 + ne)$ where $e = |E(G)|$. This algorithm has, in fact, calculated the distances among all pairs of vertices. The problem of finding all distances is a well-studied problem in graph algorithms (see [54] in a more general setting.) M. L. Fredman [37] has an $O(n^3(\log\log n/\log n)^{1/3})$ algorithm for finding dis-

D / k	2	3	4	5	6	7	8	9	10
3	10	20	38	70	128	180	286	462	708
4	15	40	95	364	731	856	2 873	4 352	13 056
5	24	66	174	532	2 734	3 420	7 800	15 360	52 224
6	32	105	336	992	7 817	13 056	32 256	68 520	188 370
7	50	122	490	1 550	10 546	35 154	93 744	296 856	1 215 672
8	57	200	807	2 808	39 223	70 308	234 360	820 280	3 984 120
9	74	585	1 248	5 150	74 906	215 688	666 672	3 019 832	15 686 400
10	91	650	1 755	10 000	132 869	486 837	1 764 720	7 714 494	47 059 200
11	94	715	3 200	14 625	158 864	898 776	4 044 492	21 345 930	179 755 200
12	133	780	4 680	21 320	354 323	1 727 180	8 370 180	48 493 900	466 338 600
13	136	845	6 560	33 345	531 440	2 657 340	10 257 408	72 541 560	762 616 400
14	183	910	8 200	51 240	804 481	9 200 460	29 762 208	164 755 080	1 865 452 680
15	186	1 215	11 712	58 560	1 417 248	7 086 240	35 947 392	282 740 976	3 630 989 376
16	197	1 600	14 640	132 496	1 771 560	14 882 658	86 882 544	585 652 704	7 394 669 856

tances of all pairs, which is faster than the straightforward algorithm for high density graphs.

The problem of finding the diameter of a graph is just to find the furthest pair of vertices while pairs with small distances can be ignored. To take advantage of this, we can use the matrix multiplication algorithm to reduce the running time, as follows.

Let A denote the adjacency matrix of G, i.e., $A = (a_{ij})$ is an $n \times n$ a matrix with $a_{ij} = 1$ if and only if $\{v_i, v_j\}$ is an edge. It is not difficult to see that in the k^{th} matrix product $\bar{A}^k = (A + I)^k$, the (i,j)-entry is nonzero if and only if there is a walk of length at most k from v_i to v_j. Therefore, the diameter $D(G)$ is the least integer D with the property that $\bar{A}^D$ has all entries nonzero. The current champion for matrix multiplication is due to an algorithm of D. Coppersmith and S. Winograd [29] and requires running time $O(n^{2.376})$. The time required in computing $D(G)$ for G is then no more than

$O(n^{2.376} \log D)$, since we can first find the least integer k so that $\overline{A}^{2^k}$ has only nonzero entries by $n^{2.376} \log D$ steps and then use binary search to determine D in another $n^{2.376} \log D$ steps.

For the complexity lower bound, since every vertex and edge must be examined to determine the diameter, the obvious lower bound is $n+e$. There is, of course, the problem of further narrowing the gap between the upper and lower bounds on the complexity of determining $D(G)$.

Problem 6.1: Find a fast algorithm for determining the diameter of a graph.

In particular, it would be of interest to find an $n^{3-\epsilon}$ algorithm without using matrix multiplication. It is easy to show that the diameter of a tree can be determined in $2n$ steps. It would also be of interest to find an efficient algorithm for determining the diameter of planar graphs.

Problem 6.2: Find an $o(n^2)$ algorithm for determining the diameter of a planar graph.

Instead of finding the diameter of G, George and Lin [39] asked the question of finding a pair of pseudoperipheral vertices, i.e., a pair $\{x, y\}$ of vertices such that $d(x, y) \geq d(v, y)$ and $d(x, y) \geq d(x, v)$ for any vertex v in G. A greedy algorithm for this problem can be described as follows:

Step (1) Start from any vertex v and find a farthest vertex u from v.

Step (2) If v is also furthest from u, $\{u, v\}$ is a solution. Stop. If there is a vertex w with $d(u, w) > d(u, v)$, go to (1) and replace v by w.

One interesting question is the following.

Problem 6.3: Is it true that this greedy algorithm must stop before $c\sqrt{n}$ steps?

There exists a graph together with a starting point such that the greedy algorithm takes $c\sqrt{n}$ iterations (see [49]). We remark that J. K. Pachl has another algorithm for solving this problem with worst case time $\sqrt{ne}$. (Note that each iteration in the greedy algorithm takes e steps). However, the complexity of the greedy algorithm still remains open.

7. Variations and Generalizations

There are many interesting variations of the diameter problem. A natural invariant of a graph is the average distance, which is defined to be the average value of the distances among all $\binom{n}{2}$ pairs of vertices (see [30,40,50]). In a network model, the average distance corresponds to the average delay whereas the diameter concerns the maximum delay.

Winkler raised several prolems of average distance bounds for altered graphs [56]. Recently, Beinstock and Györi [7] solved one of Winkler's problems by proving that for any 2-edge-connected graph there is an edge whose removal will result in a graph with average distance at most 4/3 of the average distance in the old graph. The author has studied the relation of average distance and other graph invariants and showed, for example, that the independence number is no more than the average distance [26].

Another direction is to study several disjoint short paths instead of the shortest path between two vertices. In many network models it often occurs that the messages are sent from one point to another using many possible routes.

We define the k-distance $d_k(u, v)$ between two vertices u and v to be the minimum average length of k edge disjoint paths joining u and v, if k such paths exist. The k-diameter is the maximum k-distance over all pairs of vertices. A particularly interesting case is the flow diameter. For two vertices u and v, we define the flow distance $d^*(u, v)$ to be the k-distance where k is the maximum number of edge-disjoint paths joining u and v. Similarly, we can define the flow diameter to be the maximum $d^*(u,v)$ among all pairs of vertices u and v. The notions of k-diameters and flow diameter which arise in many communication problems and in distributed computing certainly lead to a rich area for further research.

We can also consider diameters in algebraic structures such as groups (or fields and algebras). For a group with a fixed set of generators, the diameter is the minimum number m such that every group element can be represented as a product of at most m generators. Clearly the diameter of a group is just the diameter of its Cayley graph. Such graphs are often very useful in explicit constructions. The following problem arises from one such contruction:

Problem 7.1: Consider $GF(p^t)$ for some prime number p. Every element in $GF(p^t)$ can be viewed as a polynomial in x of degree $< t$ and with coefficients in $GF(p)$. That is, $GF(p^t) \simeq GF(p)[x]/(F(x))$ for some irreducible polynomial $F(X)$ of degree k. The problem is to determine minimum m such that every element in $GF(p^t)$ can be written as a product

$$(x + a_1) \cdots (x + a_m) \quad \text{for } a_i \in GF(p) \quad .$$

For sufficiently large p, is it true that $t + 1$ linear terms are enough?

Very recently, it has been shown [28] that every element in $GF(p^t)$ can be written as a product of no more than $2t + 1$ linear terms.

Many problems in self-adjusting data structure and games can be viewed as diameter problems. Namely, each configuration corresponds to a vertex. Each possible "move" corresponds to an edge. Diameters of such a graph provide exactly the worst case bound. Some related references can be found in [54].

There are many analogous problems for directed graphs or special graphs. There is also the vertex-deletion version for the diameter bounds. However, those problems will not be covered here.

Of course, we have barely scratched the surface of the possible diameter problems here. Most old problems, extremal and algrithmic, are still far from being resolved. On the other hand, interesting new problems are constantly arising. Clearly, this fertile area presents an open-ended challenge to mathematical researchers.

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