Separator Theorems and Their Applications

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1. Introduction

Two subsets U and V of vertices in a graph G are said to be *separated* if no vertex in U is adjacent to any vertex in V. A subset S of vertices in a graph G is said to be a *separator* of G if by removing vertices in S from G, the remaining graph can be partitioned into two separated parts, say A and B, satisfying $|A| \leq |B| < 2|A|$. The concept of separators has been an extremely useful tool for dealing with many families of graphs (such as trees, planar graphs). For graphs with small separators, efficient and systematic methods can be developed for solving extremal and computational problems in so-called "divide-and-conquer" fashion. Namely, the original problem is divided into two or more smaller problems. The subproblems are solved by applying the method recursively, and the solutions to the subproblems are combined to give the solution to the original problem.

There are many different formulations and variations of separator theorems scattered about in the literature. For example, some very useful separator properties involve the trade-off of the separator size and the ratio of the two separated parts as well as additional requirements when the vertices are colored. We here intend to briefly survey various separator theorems. Then we will discuss some applications of these separator theorems to an extremal graph problem of finding optimal universal graphs. We then include references to many other applications in algorithmic design, data structure, and circuit complexity as well as in a number of other areas.

2. Separator Theorems for Trees

For a subset S of vertices in a graph G, we say S separates a subgraph H in G if there is no edge between vertices in H and vertices $V(G) - S - V(H)$.

We will start from the easiest but most fundamental separator theorems (see [CG1]).

Theorem 2.1. *Suppose α is any real number more than $\frac{1}{2}$ and T is a tree with at least $\alpha+1$ vertices. Then some vertex v separates a forest F in T satisfying*

$$\alpha \leq |V(F)| < 2\alpha.$$

Proof. If $\alpha \le 1$, we choose F to be an end vertex in T which can be separated by removing one vertex. We assume $\alpha > 1$. If $|V(T)| \le \alpha + 1$, the result is immediate. We may assume $|V(T)| > \alpha + 1$. Choose a leaf v_0 in T and let $\{v_0, v_1\}$ denote the edge incident to v_0. If all the connected components of $T - v_1$ have no more than α vertices, then by taking unions of some connected components, the desired forest F can be formed. Thus we may assume that some connected component T_1 has more than α vertices. If T_1 has fewer than 2α vertices, then we take $F = T_1$. We may assume $|V(T_1)| \ge 2\alpha$. Let v_2 be the vertex in T_1 adjacent to v_1 and consider the set S_1 of connected components in $T - v_2$ not containing v_1. The total number of vertices in S_1 is at least $2\alpha - 1 > \alpha$. As before, if all trees in S_1 have no more than α vertices, then F can be formed similarly. If some tree in S_1 has at least α but fewer than 2α vertices, again we are done. If some tree T_2 in S_1 has more than 2α vertices, then we let v_3 denote the vertex in T_2 adjacent to v_2 and we consider the connected components of $T - v_3$ not containing v_2, etc. By continuing in this matter, the theorem follows by induction. □

As an immediate consequence of Theorem 2.1, we have the following (also see [CG1, LSH]).

Corollary 2.1. *Any n-vertex tree can be divided into two separated parts, each with no more than $\frac{2}{3}n$ vertices, by removing one vertex.*

Remark 2.1. If we replace "tree" by "forest", Theorem 2.1 and Corollary 2.1 are obviously true.

Remark 2.2. If we limit ourselves to binary trees, then we can achieve the same separation by removing one edge as follows:

Corollary 2.2. *For any real number $\alpha > \frac{1}{2}$, let T be a binary tree with at least $\alpha+1$ vertices. Then some edge separates a forest F in T satisfying*

$$\alpha \le |V(F)| < 2\alpha.$$

For trees with maximum degree $d+1$, the ratio of the two separated parts can be as large as d [Va2]. In a very similar way, the following can be proved.

Corollary 2.3. *For any real number $\alpha > \frac{1}{d}$, let T be a tree with maximum degree $d+1$ having at least $\alpha+1$ vertices. Then some edge separates a forest F satisfying*

$$\alpha \le |V(F)| < d\alpha.$$

If we allow removing more than one vertex from a tree, the ratio of the two separated parts can be closer to $\frac{1}{2}$. In [CGP], it was shown that by removing w vertices, the ratio of two separated parts is no more than $1 + (\frac{2}{3})^{w-1}$. We here give an improved version.

Theorem 2.2. *Let w denote a positive integer and T denote a tree with at least $\beta + w$ vertices where β is a real value more than $\frac{3^w}{2}$. Then some set of w vertices separates a forest F in T satisfying*

$$||V(F)| - \beta| \leq \left(\frac{1}{3}\right)^w \beta.$$

Proof. For w=1, this is an immediate consequence of Theorem 2.1 by choosing α to be $\frac{2}{3}\beta$. Suppose it holds for any w', where $1 \leq w' < w$. By induction we can choose a set W' of w-1 vertices so that there is a forest F_1, formed by taking the union of some connected components in $T - W'$ satisfying

$$||V(F_1)| - \beta| \leq \left(\frac{1}{3}\right)^{w-1} \beta.$$

If $||V(F_1)| - \beta| \leq \left(\frac{1}{3}\right)^w \beta$, then Theorem 2.2 holds for w. We may assume

$$\left(1 - \left(\frac{1}{3}\right)^{w-1}\right)\beta \leq |V(F_1)| < \left(1 - \left(\frac{1}{3}\right)^{w}\right)\beta$$

or

$$\left(1 + \left(\frac{1}{3}\right)^{w}\right)\beta < |V(F_1)| \leq \left(1 + \left(\frac{1}{3}\right)^{w-1}\right)\beta.$$

We consider the following two cases.

Case 1.

$$\left(1 - \left(\frac{1}{3}\right)^{w-1}\right)\beta \leq |V(F_1)| < \left(1 - \left(\frac{1}{3}\right)^{w}\right)\beta.$$

Let F_2 be the forest formed by taking the union of all the connected components in $T - W' - F_1$. Let $\beta_1 = \frac{2}{3}(\beta - |V(F_1)|)$ and apply induction assumption for F_2 if $\beta_1 > \frac{1}{2}$. There is a vertex v so that a forest F_3 can be formed by taking the union of some connected components in $F_2 - v$ satisfying

$$\beta_1 \leq |V(F_3)| < 2\beta_1.$$

Let F_4 denote the forest in $T - W' - v$ which is the union of F_1 and F_3. Then we have

$$\begin{aligned}
|V(F_4)| &= |V(F_1)| + |V(F_3)| \\
&\geq |V(F_1)| + \beta_1 \\
&\geq |V(F_1)| + \frac{2}{3}(\beta - |V(F_1)|) \\
&\geq \frac{2}{3}\beta + \frac{1}{3}|V(F_1)| \\
&\geq \frac{2}{3}\beta + \frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^{w-1}\right)\beta \\
&\geq \left(1 - \left(\frac{1}{3}\right)^{w}\right)\beta.
\end{aligned}$$

On the other hand,

$$\begin{aligned}|V(F_4)| &\le |V(F_1)| + |V(F_3)| \\ &\le |V(F_1)| + 2\beta_1 \\ &\le |V(F_1)| + \frac{4}{3}(\beta - |V(F_1)|) \\ &\le \frac{4}{3}\beta - \frac{1}{3}|V(F_1)| \\ &\le \frac{4}{3}\beta - \frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^{w-1}\right)\beta \\ &\le \left(1 + \left(\frac{1}{3}\right)^{w}\right)\beta.\end{aligned}$$

Suppose $\beta_1 \le \frac{1}{2}$. Then $\beta - V(G_1) \le \frac{3}{4}$. We choose a vertex v' so that there is an isolated vertex v'' in $T - W' - F_1 - v'$. Let F_4' be the forest in $T - W' - v'$ which is the union of F_1 and v''. We have

$$|V(F_4')| = |V(F_1)| + 1 \ge \beta + \frac{1}{4} \ge \beta \quad \text{and}$$

$$\begin{aligned}|V(F_4')| = |V(F_1)| + 1 &\le \left(1 - \left(\frac{1}{3}\right)^{w}\right)\beta + 1 \\ &\le \left(1 + \left(\frac{1}{3}\right)^{w}\right)\beta.\end{aligned}$$

We can then take F to be F_4 or F_4'.

Case 2.

$$\left(1 + \left(\frac{1}{3}\right)^{w}\right)\beta < |V(F_1)| \le \left(1 + \left(\frac{1}{3}\right)^{w-1}\right)\beta.$$

Let $\beta_2 = \frac{2}{3}(|V(F_1)| - \frac{3}{4} - \beta)$ and apply the induction assumption for F_1 if $\beta_2 > \frac{1}{2}$. There is a vertex u so that a forest F_5 can be formed by taking a union of some connected components in $F_1 - u$ satisfying

$$\beta_2 \le |V(F_5)| < 2\beta_2.$$

Let F_6 denote the forest $F_1 - F_5 - u$ which is the union of some connected components in $T - W' - u$. We then have

$$\begin{aligned}|V(F_6)| &= |V(F_1)| - |V(F_5)| - 1\\ &\geq |V(F_1)| - 2\beta_2 - 1\\ &\geq |V(F_1)| - \frac{4}{3}(|V(F_1)| - \frac{3}{4} - \beta) - 1\\ &\geq \frac{4}{3}\beta - \frac{1}{3}|V(F_1)|\\ &\geq \frac{4}{3}\beta - \frac{1}{3}\left(1 + \left(\frac{1}{3}\right)^{w-1}\right)\beta\\ &\geq \beta - \left(\frac{1}{3}\right)^{w}\beta.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}|V(F_6)| &= |V(F_1)| - |V(F_5)| - 1\\ &\leq |V(F_1)| - \frac{2}{3}(|V(F_1)| - \frac{3}{4} - \beta) - 1\\ &\leq \frac{1}{3}|V(F_1)| + \frac{2}{3}\beta - \frac{1}{2}\\ &\leq \frac{1}{3}\left(1 + \left(\frac{1}{3}\right)^{w-1}\right)\beta + \frac{2}{3}\beta\\ &\leq \beta + \left(\frac{1}{3}\right)^{w}\beta.\end{aligned}$$

Suppose $\beta_2 \leq \frac{1}{2}$. Then $|V(F_1)| - \frac{3}{4} - \beta \leq \frac{3}{4}$. We can choose a vertex u' so that there is an isolated vertex u'' in $F_1 - u'$. Let F_6' denote the forest $F_1 - u' - u''$ which is the union of some connected components in $T - w' - u'$. We have

$$|V(F_6')| = |V(F_1)| - 1 \leq \beta + \frac{1}{2} \leq \left(1 + \left(\frac{1}{3}\right)^{w}\right)\beta$$

$$\begin{aligned}|V(F_6')| = |V(F_1)| - 1 &\geq \left(1 + \left(\frac{1}{3}\right)^{w}\right)\beta - 1\\ &\geq \left(1 - \left(\frac{1}{3}\right)^{w}\right)\beta.\end{aligned}$$

F_6 or F_6' is the forest F we want. This completes the proof for Theorem 2.2.

Corollary 2.4. *Any n-vertex tree can be divided into two separated parts, each with no more than $\left(1 + \left(\frac{1}{3}\right)^{w}\right)\frac{n}{2}$ vertices, by removing w vertices, if $n > 3^w$.*

Theorem 2.3. *Any n-vertex tree can be divided into two separated equal parts by removing at most $\lfloor \frac{\log n}{\log 3} \rfloor + 1$ vertices.*

Proof. Using Theorem 2.2 and setting $w = \lfloor \log_3(n - \log_3 n) \rfloor$, $\beta = \frac{(n - \log_3 n)}{2}$, it is easily checked that $\beta \cdot (\frac{1}{3})^w < \frac{3}{2}$. Therefore by removing $w+1$ vertices, the tree is divided into two separated equal parts. □

A separator is called a bisector if it separates the graph into two equal parts. In [BCG], examples have shown that the bisectors in Theorem 2.3 are very close to the optimum.

Theorem 2.4. *In the complete ternary tree on 2t levels with* $n = \frac{3^{2t}-1}{2}$ *vertices, one cannot remove fewer than* $\frac{\log n}{\log 3} - \frac{2 \log n}{\log \log n}$ *vertices to separate the remaining graph into two equal parts.*

Remark 2.3. Theorems 2.1 and 2.2 can all be generalized by considering a cost assignment to vertices instead of counting vertices. Here we state the generalized versions whose proofs are extremely similar and will not be given.

Theorem 2.1′. *Let F be any n-vertex forest with non-negative vertex costs. Let α be any real number that is greater than half of the maximum cost of the vertices and is smaller than the sum of the total vertex costs subtracting the minimum vertex cost. Then some vertex v separates a forest F' so that the total cost of vertices in F' is between α and 2α.*

Theorem 2.2′. *Let w denote a positive integer and F denote a forest with non-negative vertex costs. Let β be a real value that is at least $\frac{3^w}{2}$ times the maximum vertex cost and is smaller than the sum of the all vertex costs except for the w smallest vertex costs. Then some set of w vertices separates a subforest F' and the total cost $c(F')$ of vertices in F' satisfies*

$$|c(F') - \beta| \leq \left(\frac{1}{3}\right)^w \beta.$$

3. Separators for Colored Trees

Suppose the vertices of a tree are colored in k colors for some given integer k. It is often desirable to remove a small number of vertices so that the remaining graph is separated into two parts, each having about a half of the number of vertices in each color. This has the similar flavor of a very nice result of Goldberg and West [GW].

Theorem 3.1. *Suppose the beads on a string are colored in k colors. One can cut the necklace at k places so that the resulting strings of beads can be placed into two piles, each of which contains the equal (to within one) number of beads in each color.*

This necklace-splitting theorem can be combined with a decomposition lemma to derive the tree-splitting analogue (see [BL]). Since the subgraphs of a tree inherently have small separators, we can associate the separators in a complete binary tree structure. A complete binary tree of t levels has a vertex set consisting of all (0,1)-tuples of length $< t$ together with the root denoted by *. Each vertex u is adjacent to its two children $u0$ and $u1$.

Lemma 3.1. *Any n-vertex tree can be mapped into a complete binary tree C of* $\lceil \frac{\log n}{\log 3/2} \rceil$ *levels satisfying the following properties.*

(i) The mapping f is 1-1 from V(G) into V(C).
(ii) For each w in C, let S_w denote the set of all vertices in T that are mapped to descendants of w in C. Then $|S_{w0}| \leq |S_{w1}| < 2|S_{w0}|$.
(iii) For each w in C, let A_w denote the set of all vertices in T that are mapped to the ancestors of w in C and w itself. Then by removing A_w, the set S_w is separated from the rest of the graph in $T - A_w$.

Proof. This follows immediately by mapping the separator vertex of Theorem 2.1 to the root and the two separated parts to S_0 and S_1, recursively.

This decomposition tree induces a natural linear order for vertices in T which can be viewed as the order from left to right in the plane layout of C. To be precise, for two distinct binary tuples α and β of length $\leq t$, we say $\alpha < \beta$ if at the first place they differ, the corresponding coordinate of α is 0 and the corresponding place for β is 1.

Lemma 3.2. *Each initial segment, which consists of all u with $f(u) < w$ for some w in C, can be separated from the rest of the tree by removing vertices that mapped to the path $A(w)$ consisting of all ancestors of w.*

Combining Theorem 3.1 and Lemmas 3.1 and 3.2, we can then arrive at the following result for splitting colored trees (see [BL]).

Theorem 3.2. *Suppose the n vertices of a tree are colored in k colors. By removing $ck \log n$ vertices, the connected components can be partitioned into two parts, each containing the same number of vertices in each color.*

Proof. First we use Lemma 3.1 to decompose the tree T. That is to map the vertices of T into a complete binary tree C of $\lceil \frac{\log n}{\log 3/2} \rceil$ levels. Using the linear order introduced by the decomposition tree, the colored vertices can be viewed as a string of beads, so we are ready to use the necklace-splitting theorem. Theorem 3.1 ensures that by making no more than k cuts in the string, the k-colored beads can be almost equally partitioned into two parts. Now, for each cut, choose a vertex w next to the cut and let S_0 consist of all vertices in the path $A(w)$ consisting of all ancestors of w. It follows from Lemma 3.2 that by removing all vertices in S_0, the remaining graph in T can be partitioned into two parts so that the sum of differences of the number of vertices in each color is no more than $|S_0| + k$. By adding additional $|S_0| + k$ vertices to S_0 to form the separator S, the number of vertices of each color in the two parts can then be balanced. The separator S contains no more than $2k(\lceil \frac{\log n}{\log 3/2} \rceil + 1)$ vertices. This completes the proof of Theorem 3.2.

Although Theorem 3.2 is best possible within a constant factor, there is a trade-off between the size of the separator and the precision of bipartition. This is often crucial in proving optimality of various universal graphs in later sections [BCLR1, BCLR2].

Theorem 3.3. *For each constant $p < \frac{1}{2}$, there exists a constant q such that any n-vertex forest with w vertices of color A can be partitioned into two sets by the removal of q vertices so that each set has at least $\lfloor pn \rfloor$ vertices and at least $\lfloor pw \rfloor$ vertices of color A.*

The proof of Theorem 3.3 mainly follows from forming an appropriate decomposition tree and will not be included here. There is a generalized version that will be stated here without proof.

Theorem 3.4. *For each constant $p < \frac{1}{2}$ and each positive integer k, there exists a constant q such that any n-vertex forest with n_i vertices of color c_i, $i = 1, \ldots, k$, can be partitioned into two sets by the removal of q vertices so that for each i, each set has at least $\lfloor p(n_1 + \ldots + n_i) \rfloor$ vertices in color $1, 2, \ldots i$.*

4. Separator Theorems for Planar Graphs

Separator theorems for planar graphs and their applications were first described in the seminal papers of Lipton and Tarjan [LT1, LT2]. Here we will state without proof several major versions.

Theorem 4.1 [LT1]. *Let G be any n-vertex planar graph with non-negative vertex costs. Then the vertices of G can be partitioned into three sets A, B, C, such that no edge joins a vertex in A with a vertex in B, neither A nor B has vertex cost exceeding $\frac{2}{3}$ of the total cost, and C contains no more than $2\sqrt{2}\sqrt{n}$ vertices.*

Remark 4.1. The constant $2\sqrt{2}$ was improved by Djidjev [D1] to $\sqrt{6}$ and later by Gazit [G] to $\frac{7}{3}$.

Theorem 4.2 [LT1]. *Let G be any n-vertex planar graph with non-negative vertex costs. Then vertices of G can be partitioned into three sets A, B, C, such that no edge joins a vertex in A with a vertex in B, neither A nor B has vertex cost exceeding $\frac{1}{2}$ of the total cost, and C contains no more than $2\sqrt{2}/(1 - \sqrt{2/3})\sqrt{n}$ vertices.*

Remark 4.2. The constant $2\sqrt{2}/(1-\sqrt{2/3}) \approx 15.413$ was later improved in several papers; $\sqrt{6}/(1 - \sqrt{2/3}) \approx 13.348$ in [D1] and $8 + \frac{16}{81}\sqrt{6}/(1 - \sqrt{2/3}) \approx 10.637$ in [Ve]. The current best constant is $\frac{108}{19}\sqrt{3} \approx 9.845$ (see [C1]). On the other hand, the best known constant for the lower bound is $\frac{2}{\sqrt{3}} \approx 1.155$ by considering a plane graph with vertex set consisting of points $\{(x, y\sqrt{3/2}) : x, y \in z$ and $(x, y\sqrt{3/2})$ is in a regular hexagon of side $\sqrt{n/3}\}$ and the edges are between points of distance at most 1.

Remark 4.3. The separator can be constructed in $O(n)$ time [LT1].

The constant can be further improved if we consider the separation of a planar graph into two parts, each with roughly equal number of vertices. In other words, every vertex has a cost of 1.

Theorem 4.3 [C1]. *Let G be any n-vertex planar graph. The vertices of G can be partitioned into three sets A, B, C, such that no edge connects A with B, A and B each have $\leq \frac{n}{2}$ vertices, and C contains $\leq 3\sqrt{6}\sqrt{n}$ vertices.*

The constant $3\sqrt{6} \approx 7.348$ is an improvement over the results in [LT1], [D1] and [Ve].

We will give the proof for Theorem 4.3 which needs the following facts.

Lemma 4.1 [LT1]. *Let G be a planar graph of radius s, with non-negative vertex-costs summing to 1. Then the vertices of G can be partitioned into three parts A, B, C, such that there is no edge between A and B, neither A nor B has total vertex cost exceeding $\frac{2}{3}$, and C contains at most $2s+1$ vertices.*

Lemma 4.2 [D1]. *Let G be an n-vertex planar graph of radius s. For any real number r, $\frac{1}{2} \leq r \leq 1$, there exists a set $S \subseteq V(G)$ with at most $3s+1$ vertices such that by removing vertices in S from G the remaining graph is separated into three parts A, B, C, such that A, B each contain at most $(1-r)n$ vertices, and C contains at most rn vertices.*

Lemma 4.3 [Ve]. *For any integer s, an n-vertex planar graph G contains a subgraph H of at least $n-\frac{n}{s}$ vertices so that any subgraph of H can be embedded into another planar graph of radius $s-1$.*

Proof of Theorem 4.3. Let G denote a planar graph on n vertices. We will determine A, B and C iteratively as follows:

Step 0. Set $s = \lfloor\sqrt{\frac{n}{6}}\rfloor$ and use Lemma 4.3 to find a set S_0 with at most $\frac{n}{s}$ vertices and embed $G - S_0$ into a graph G' of radius $s-1$. Set $A = B = \emptyset$, $C = S_0$ and $j = 1$.

In general, for $i = 1, 2, \ldots$, the step i can be described as follows.

Step i. Set $r = (\frac{n}{2} - |A|)/n'$ where $n' = |V(G')|$. For $j \leq 2$, use Lemma 2 to find a separator S' containing at most $3(s-1)$ vertices which separate G' into A', B' and C' with $|A'| \leq |B'| \leq (1-r)n'$ and $|B'| \leq |C'| \leq rn'$. Set A to be the smaller one of $A \cup C'$ and $B \cup B'$; B to be the larger one of $A \cup C'$ and $B \cup B'$; and C to be $C \cup S'$. If $j < 2$, apply Lemma 3 to the induced subgraph of A' and form a new graph G' of radius $s-1$, set $j = j+1$ and repeat Step i. If $j = 2$, set i to be $i+1$, j to be 1. Then set s to be $\sqrt{n'/6}$ when applying Lemma 3 to A' and form G' of radius $s-1$, add $\frac{n'}{s}$ vertices to C. Then go to the next step until G' is empty.

The correctness of this algorithm can be established by verifying the following facts in Step i. Suppose at the beginning of Step i A has p vertices, B has

q vertices and G' has w vertices. The new A and B have no more than the maximum of $p+|C'|$ and $q+|B'|$. Since $p+|C'| \le p+rw \le p+(\frac{n}{2}-p) \le \frac{n}{2}$ and $q+|B'| \le q+(1-r)w \le q+w-\frac{n}{2}+p \le \frac{n}{2}$, the new A and B has no more than $\frac{n}{2}$ vertices. Furthermore $|A'| \le \frac{w}{3}$ since $|A'| \le |B'| \le |C'|$ and $|A'|+|B'|+|C'| \le w$. This implies the new G' has at most $\frac{w}{3}$ vertices. Because of our choice of j, for each i, Step i is repeated twice for $j=1$ and 2 (except for possibly the last step). So altogether $|G'|$ is reduced by a factor of 9.

We can bound the separator $C = C(n)$ as follows:

$$\begin{aligned} C(n) &\le 2\sqrt{6}\sqrt{n} + \sum_{i\ge 1} 2\sqrt{6}\sqrt{\frac{n}{9^i}} \\ &\le 2\sqrt{6}\sqrt{n} + \left|C\left(\frac{n}{9}\right)\right|. \end{aligned}$$

Therefore

$$|C(n)| \le 3\sqrt{6}\sqrt{n}.$$

This completes the proof for Theorem 4.3.

Miller [M] further requires the planar separator to be a simple cycle.

Theorem 4.4 [M]. *If G is an embedded 2-connected planar graph with non-negative weights assigned to vertices and faces that sum to 1, and no face has weight $> \frac{2}{3}$, then there exists a simple cycle on at most $2\sqrt{2n}$ vertices so that neither the interior nor the exterior has total weight $> \frac{2}{3}$.*

For graphs with genus g, the separator is of size $c\sqrt{gn}$ [D2, GHT]. The best constant is given in [D3].

Theorem 4.5. *Any n-vertex graph of orientable genus g has a separator with $\sqrt{2g+1}\sqrt{6n}$ vertices.*

5. Separator Theorems and Universal Graphs

Separator theorems have many applications in a broad range of areas. In this section, we will illustrate some applications in extremal graph theory, namely in the construction of universal graphs.

The problem of universal graphs is a fundamental problem that arises in various contexts in many topics such as universal circuit [Va2], data representation [CRS, RSS], VLSI design [CR, BCG] and simulations of parallel computer architecture [BCLR1, BCHLR]. A typical problem is the following:

How many edges must a graph have that contains all trees with n vertices?

Obviously, the complete graph on n vertices and $\binom{n}{2}$ edges has the required universal property. However, the objective is to determine the minimum number $f(n)$ of edges in such a universal graph $G(n)$, which contains all n-vertex trees.

It is easy to see that the universal graph $G(n)$ must contain at least $\frac{1}{2}n \log n$ edges since it must contain one vertex of degree $\geq n-1$, two vertices of degree $\geq \frac{n}{2}$, and, in general, i vertices of degree $\geq \frac{n}{i}$ so that its degree sequence dominates $n-1, \frac{n}{2}, \frac{n}{3}, \ldots$. On the upper bound for $f(n)$, we can improve upon $\binom{n}{2}$ of the complete graph by the following series of applications of the separator theorem for trees.

Construction 1. $G_1(n)$ is the union of $G_1\left(\frac{2}{3}n\right)$ and $G_1\left(\frac{n}{2}\right)$ together with a vertex u that is adjacent to all other vertices. By using Theorem 2.1, any n-vertex tree can

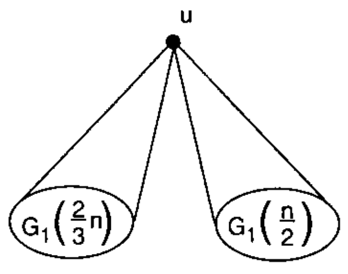


Fig. 1

be embedded into $G_1(n)$ by mapping the separator to u and the separated parts to $G_1\left(\frac{2}{3}n\right)$ and $G_1\left(\frac{n}{2}\right)$ respectively. On the other hand, the number of edges in $G_1(n)$ is bounded above by $f_1(n)$, which satisfies the following inequality[1]:

$$f_1(n) \geq f_1\left(\frac{2}{3}n\right) + f_1\left(\frac{n}{2}\right) + n.$$

It is easy to verify that

$$|E(G_1(n))| \leq f_1(n) \leq cn^{1.3}.$$

Although this is significantly better than the complete graph, it can be much improved by using Theorem 2.3.

Construction 2. $G_2(n)$ is the union of two copies of $G_2\left(\frac{n}{2}\right)$ together with $\frac{\log n}{\log 3}$ vertices, each of which is adjacent to all other vertices. A straightforward application of Theorem 2.3 shows that $G_2(n)$ contains all trees on n vertices and the number of edges in $G_2(n)$ is bounded by $f_2(n)$ which satisfies

$$f_2(n) \geq 2f_2\left(\frac{n}{2}\right) + n \log_3 n.$$

[1] Strictly speaking, we should use $G_1(\lfloor\frac{2}{3}n\rfloor)$ and $G_1(\lfloor\frac{n}{2}\rfloor)$. However, we will usually not bother with this type of detail since it has no significant effect on the arguments or results.

It can be easily checked that

$$|E(G_2(n))| \leq f_2(n) \leq cn\log^2 n.$$

The above bound can be further improved by using Theorem 2.2 appropriately (see [CGP]).

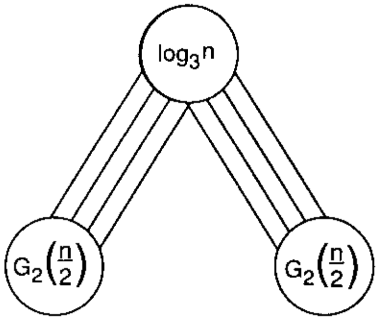


Fig. 2

Construction 3. $G_3(n)$ consists of $G_3(m)$ and $G_3(m)$ together with w vertices that are adjacent to all other vertices. By choosing $w = \frac{\log\log n}{\log 3/2}$ and

$$m = w \cdot n$$

we can then derive

$$|E(G_3(n))| \leq n\log n(\log\log n)^2.$$

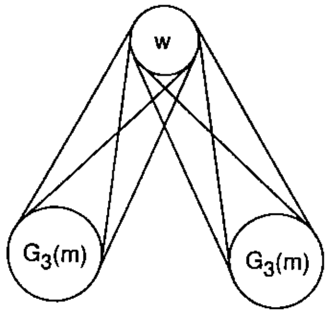
In fact, it has been shown [CG3] that the minimum number $f(n)$ of edges in a graph that contains all n-vertex trees satisfies

$$\frac{1}{2}n\log n < f(n) < \frac{7}{\log 4}n\log n.$$

The construction assumption and proof are based on an elaborated induction and repeated usage of Theorem 2.1 that we will not discuss here. Our relatively simple descriptions of construction 1-3 merely illustrate various ways of using separator theorems and their effects.

A related problem is to determine the minimum number $f(n,d)$ of vertices in a graph that contains all n-vertex trees with maximum degree d. This problem can be solved by using the separator theorems for colored trees (Theorems 3.2, 3.3) and the decomposition lemmas. We here consider a simpler version of universal graphs for binary trees, that are trees with maximum degree 3 (see [BCLR2]).

Lemma 5.1. *The n vertices of a binary tree T can be mapped into a complete binary tree C on no more than $2^q - 1$ vertices ($2^q - 1 \leq n < 2^{q+1} - 1$) so that $6\log\frac{n}{2^t} + 18$ vertices of T are mapped into a vertex of C at distance t from the root, and so that any two vertices adjacent in T are mapped to vertices at most 3 apart in C.*



$G_3(n)$

Fig. 3

Proof. The idea is to recursively bisect T, placing the successive sets of bisector vertices within successively lower levels of C, until T is decomposed into single vertices. For example, the vertices placed at the root of C bisect T into two subgraphs T_1 and T_2. Similarly, vertices mapped to the left child of the root bisect T_1 and vertices mapped into the right child bisect T_2. In addition, at level i of C we map vertices of T (that have not already been mapped within levels $i-1$, $i-2$) that are adjacent to vertices mapped at level $i-3$ of C. This ensures that vertices adjacent in T will be mapped to vertices of C at most distance 3 apart.

To keep the number of vertices of T mapped to a level i vertex in C within the required bounds, we use separators for 3 colors, so-called 3-bisectors, as in Theorem 3.2. The following procedure describes how this is done.

Step 0. Initialize every vertex of T to color A, bisect T, and place the bisector vertices at the root (level 0) of C.

Step 1. For each subgraph created in the previous step, recolor every vertex adjacent to the bisector in the previous step with color 0, and place a 2-color bisector for the subgraph at the corresponding level 1 vertex of C.

Step 2. For each subgraph created in the previous step, recolor every vertex of color A adjacent to the bisector in the previous step with color 1, and place a 3-color bisector for the subgraph at the corresponding level 2 vertex of C.

Step t. ($\log|T| \geq t \geq 3$). For each subgraph created in the previous step, place every vertex of color $t-1$ (mod 2) at the corresponding level t of C, recolor every vertex of color A that is adjacent to one of color $t-1$ (mod 2) with color t (mod 2), and place a 3-color bisector for the remaining subgraph at the corresponding level t vertex of C.

To ensure the accuracy of Step t, it suffices to show $n_t \leq 6\log\frac{n}{2^t} + 18$ for $3 \leq t < \log|T|$. Since we have

$$n_t \le 3 \log \frac{n}{2^t} + \frac{1}{2} n_{t-3}$$

$$\le 6 \log \frac{n}{2^t} + 18,$$

Lemma 5.1 is proved.

The analogous version for bounded-degree trees and planar graphs can be proved in a very similar way and the proofs are left to the reader. The main difference in proving these results is that vertices adjacent to previously mapped vertices are themselves only mapped at every $\log d$ instead of at every level.

Lemma 5.2. *The vertices of a tree T with maximum degree d can be mapped into a complete binary tree C on no more than $2^q - 1$ vertices ($2^q - 1 \le n < 2^{q+1} - 1$) so that $o(\log \frac{n}{2^t})$ vertices of T are mapped to a vertex of C at distance t from the root, and so that any two vertices adjacent in T are mapped to vertices at most distance $O(\log d)$ apart in C.*

Lemma 5.3. *The vertices of a planar graph G of maximum degree d can be mapped into a complete binary tree C on $2^q - 1$ vertices ($2^q - 1 \le n < 2^{q+1} - 1$) so that $O(\sqrt{n/2^t})$ vertices of G are mapped to a vertex of C at distance t from the root, and so that any two vertices adjacent in G are mapped to vertices at most distance $O(\log d)$ apart in C.*

A graph G is said to have a *k-bisector function f* if any subgraph of G on m vertices has a k-bisector of size no more than $f(m)$. The preceding lemmas are all special cases of the following.

Lemma 5.4. *Suppose G on n vertices with maximum degree d has a k-bisector function f. The vertices of G can be mapped into a complete binary tree C on no more than $2^q - 1$ vertices where $2^q - 1 \le n < 2^{q+1} - 1$ so that $\alpha d f(\frac{n}{2^t})$ vertices of G are mapped to a vertex of C at distance t from the root, and so that any two vertices adjacent in G are mapped to vertices at most distance k apart in C if*

$$2f(xd^{3k}) \le d^{3k-4} f(x) \text{ for all } x.$$

Although Lemma 5.4 looks somewhat complicated, it is merely a straightforward generalization of Lemmas 5.1 ≈ 5.3, and we omit the proof. We can now construct universal graphs using the decomposition lemmas.

Theorem 5.1. *The minimum universal graph for the family of all bounded-degree trees on n vertices has n vertices and $O(n)$ edges.*

Proof. Using Lemma 5.2, we consider the graph with vertices grouped into clusters corresponding to the vertices in the complete binary tree C. A cluster corresponding to a vertex of level t contains $O(\log \frac{n}{2^t})$ vertices. We connect all pairs of vertices in clusters with corresponding vertices within distance $O(\log d) = O(1)$ apart in C. By Lemma 5.2 the resulting graph is universal for the family of all trees with maximum degree d. The number $h(n)$ of edges in this graph is $O(n)$, since $h(n)$ satisfies the following recurrence inequality:

$$h(n) \leq 2h\left(\frac{n}{2}\right) + c(\log n)^2$$

where c is an appropriate constant depending on d.

The construction just described has $O(n)$ vertices. To obtain a universal graph with precisely n vertices, we modify the embedding of Lemma 5.1 so that the same number of vertices of T are wrapped to vertices in the same level of C. This is easy to do since we can always arbitrarily expand the bisector of any subtree to be within one of its maximum allowed value (which is the lesser of the number of vertices remaining and $O(\log \frac{n}{2^t})$ for vertices on level t of C). The exact value of the maximum bisector is the same for all vertices on a level and depends on the parity of the number of vertices in the subgraphs at that level. Hence, the size of the bisectors at each level depends only on n, and the universal graph can be assumed to have precisely n vertices.

Theorem 5.2. *The minimum universal graph for the family of all bounded-degree planar graphs on n vertices has n vertices and $O(n \log n)$ edges.*

Proof. The construction is by using Lemma 5.3 in similar fashion as in the proof of Theorem 5.1. The number of edges $h(n)$ satisfies

$$h(n) \leq 2h\left(\frac{n}{2}\right) + cn$$

and therefore the minimum universal graph has $O(n \log n)$ edges. □

Theorem 5.3. *The minimum universal graph for a family of bounded-degree graphs on n vertices with bisector function $f(x) = n^\alpha$ has n vertices with: $O(n)$ edges if $\alpha < \frac{1}{2}$; $O(n \log n)$ edges if $\alpha = \frac{1}{2}$; $O(n^{2\alpha})$ edges if $\alpha > \frac{1}{2}$.*

Proof. The construction follows from Lemma 5.4 together with the fact that the number $h(n)$ of edges satisfies

$$h(n) \leq 2h\left(\frac{n}{2}\right) + (f(n))^2.$$

Theorem 5.4. *The minimum universal graph for the family of all bounded-degree outerplanar graphs on n vertices has n vertices and $O(n)$ edges.*

Proof. Since an outerplanar graph on n vertices has a bisector of size $O(\log n)$, the result follows from Theorem 5.3.

Recently, the author has shown that there is a universal graph with cn edges that contains all bounded-degree n-vertex trees as induced subgraphs [C2]. Many results on universal graphs and induced universal graphs for various classes of graphs can be found in [Bo, BCEGS, CG1, CG2, CGP, CCG, CGS, R]. In particular, the question of determining the minimum universal graphs that contain all n-vertex planar graphs remains open (it is between $n \log n$ and $n^{3/2}$ (see [BCEGS])).

6. Concluding Remarks

There are many aspects of separator theorems that we have not covered here. The references include various papers for many applications such as approximation algorithms [Ba, LT2], dynamic programming [LT2], pebbling [LT2], VLSI layout [BCLR1, BL, L], Boolean circuits [Va2], routing [LT2], nested dissection methods in numerical analysis [LRT, PR2] and parallel algorithms [GM, PR1] to find planar separators.

Very recently, N. Alon, P. Seymour and R. Thomas have generalized the separator theorem of Tarjan and Lipton to graphs excluding certain minors. A graph H is said to be a minor of a graph G if H can be obtained from a subgraph of G by contracting edges. The Kuratowski theorem asserts that a graph is planar if and only if it does not have K_5 or $K_{3,3}$ as minors. The separator theorem for graphs excluding minors can be stated as follows [AST]:

Let G be any n-vertex with non-negative vertex costs and the complete graph K_h on h vertices is not a minor of G. Then the vertices of G can be partitioned into three parts A, B and C, such that no edge joins a vertex in A with a vertex in B, neither A nor B has vertex cost exceeding $\frac{2}{3}$ of the total cost, and C contains no more than $h^{3/2}n^{1/2}$ vertices. Such a separator can be determined in time $O(h^{1/2}n^{1/2}m)$, where $m = n + |E(G)|$.

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