The Largest Minimal Rectilinear Steiner Trees for a Set of *n* Points Enclosed in a Rectangle with Given Perimeter

F. R. K. Chung
F. K. Hwang
Bell Laboratories
Murray Hill, New Jersey

ABSTRACT

Suppose P is a set of points in the plane with rectilinear distance. Let $\ell_s(P)$ *denote the length of a Steiner minimal tree for P. Let* $\ell_r(P)$ *denote the semiperimeter of the smallest rectangle with vertical and horizontal lines which encloses P. It is well known that* $\ell_s(P) \geq \ell_r(P)$ *for* $|P| \geq 3$ *where* $|P|$ *denotes the cardinality of P.*

In designing placement algorithms for printed circuits, $\ell_r(P)$ *has been used as an estimate of* $\ell_s(P)$ *when* $|P|$ *is small. Therefore, it is of some interest to know the value of*

$$\rho_n \equiv \underset{|P|=n}{Max} \frac{\ell_s(P)}{\ell_r(P)}.$$

In this paper we show ρ_n *tends to* $(\sqrt{n}+1)/2$ *and we give the exact value of* ρ_n *for* $n \leq 10$.

1. INTRODUCTION

A *rectilinear Steiner tree* (RST) for a set of points P in the plane is a tree which interconnects P using only vertical and horizontal lines. A *minimal* RST is such a tree with shortest possible total length which is denoted by $\ell_s(P)$. Minimal RST's have potential application to wire layout algorithms for printed circuits.

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It has recently been proved [2] that the construction for minimal RST's is an NP-complete problem in general. Therefore it becomes increasingly important to learn some general properties of minimal RST's. In this paper we study an extremal problem of minimal RST's. Let R(P) denote the smallest rectangle (with vertical and horizontal sides) enclosing P and let $\ell_r(P)$ denote its semiperimeter. Define

$$\rho(P) = \frac{\ell_s(P)}{\ell_r(P)}$$

and

$$\rho_n = \max_{|P|=n} \rho(P)$$

where $|P|$ is the cardinality of P. We show that $\rho_n = (\sqrt{n}+1)/2$ when n is a square. Since ρ_n is monotone nondecreasing in n, $(\sqrt{n}+1)/2$ is a reasonable estimate of ρ_n for large n. However, the problem of determining exact values for ρ_n seems to be very difficult even for not so large n. The values of ρ_n for small n are of particular interest since in the printed circuit application n is usually a single digit number. It is trivial to note that $\rho_2 = 1$. The values of ρ_3, ρ_4 and ρ_5 are given in [3]. In this paper, we determine ρ_n for $6 \le n \le 10$. The values of ρ_n for $2 \le n \le 10$ are listed in Table 1.

Table 1

n	2	3	4	5	6	7	8	9	10
ρ_n	1	1	3	3/2	5/3	7/4	11/6	2	2

$\ell_r(P)$ has been suggested and used as an estimate of $\ell_s(P)$ in [4,5]. Since $\rho(P) \ge 1$ for all P, and from Table 1, $\rho(P) \le 2$ for $n \le 10$, the error factor for such an estimate is at most two for these values of n.

2. SOME PRELIMINARY RESULTS

Let P be the set of points to be interconnected. Consider the rectilinear grid formed by drawing a horizontal and vertical line through each point of P. G(P) will denote the portion of the grid within R(P). A fundamental theorem on minimal RST's which reduces their construction to a finite problem is the following.

Hanan's Theorem [3]: *There is a minimal RST for P which is composed of a subset of the lines of G(P).*

Now we state and prove several lemmas which will be used to derive ρ_n.

Lemma 1: ρ_n *is monotone increasing in n.*

Proof: Suppose P is the set of points for which ρ_n is achieved. Let a point p be added inside of R(P). Since

$$\ell_s(P+\{p\}) \geq \ell_s(P) \quad \text{and}$$

$$\ell_r(P+\{p\}) = \ell_r(P),$$

we have

$$\rho_{n+1} \geq \frac{\ell_s(P+\{p\})}{\ell_r(P+\{p\})} \geq \rho_n.$$

Lemma 2: *Suppose* $n \geq ab$. *Then* $\rho_n \geq \frac{ab-1}{a+b-2}$.

Proof: Suppose n = ab. Consider an $a \times b$ grid where each edge is of unit length and each grid point is a point of P. Then

$$\ell_r(P) = (a-1) + (b-1).$$

But a minimal RST on P must contain at least ab - 1 edges, and hence is of length at least ab - 1. (It is easy to see that its length is indeed ab - 1.) Therefore

$$\rho_{ab} \geq \frac{\ell_s(P)}{\ell_r(P)} \geq \frac{ab-1}{(a+b-2)}.$$

Lemma 2 now follows immediately from Lemma 1.

Let $S_n(a,b)$ denote the greatest length a minimal RST for a set of points contained in a rectangle whose sides have lengths a and b can have. Recently, the following results have been proven.

Chung and Graham's Theorem [1]:

$$S_n(a,b) \le \frac{1}{2}\left(a+b+at+\frac{bn}{t}\right),$$

where t is an arbitrary positive integer.

Using Lemma 2 and this theorem, we obtain

Theorem 1: $\rho_n = \frac{\sqrt{n}+1}{2}$ *for n a square.*

Proof: By Chung and Graham's theorem we have

$$\rho_n \le \underset{a,b}{\mathrm{Max}} \frac{S_n(a,b)}{a+b} \le \underset{a,b}{\mathrm{Max}} \frac{\frac{1}{2}(a+b+at+b\;n/t)}{a+b},$$

for any positive integer t. When n is a square, let $t = \sqrt{n}$ in the above inequality. Then we have

$$\rho_n \le \frac{1}{2}(\sqrt{n}+1).$$

But from Lemma 2, for n a square

$$\rho_n \ge \frac{n-1}{2\sqrt{n}-2} = \frac{\sqrt{n}+1}{2}$$

Theorem 1 is proven.

Define d(A,B) to be the (rectilinear) distance between two points A and B which is composed of the horizontal distance $d_x(A,B)$ and the vertical distance $d_y(A,B)$.

Lemma 3: Suppose $\rho_n > k$ *where k is a positive integer. Then* $\rho(P) = \rho_n$ *implies that each side of R(P) contains at least k+1 points of P.*

Proof: Suppose on the contrary that R(P) has a side AB containing only m points $\{U_1,\ldots,U_m\}$ with $m \le k$. We may assume that all other points of P are contained in a rectangle CDEF (see Figure 1) with d(A,E) > 0.

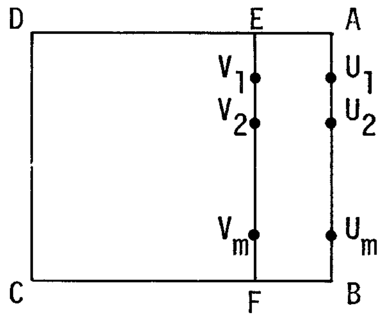

Fig. 1

Let $\{V_1,\ldots,V_m\}$ be m points on line EF such that $d_y(U_i,V_i) = 0$ for $i = 1,\ldots,m$. Let $P' = P \cup \{V_1,\ldots,V_m\} - \{U_1,\ldots,U_m\}$. Then $n \geq |P'|$.

Let t' be a minimal RST on P' and let t be an RST on P formed by adding edges (U_i,V_i), $i = 1,\ldots,m$ to t'. Since

$$\frac{\ell(t')}{\ell_r(P')} = \rho(P') \leq \rho_n \quad \text{and}$$

$$m < \rho_n ,$$

it follows that

$$\frac{\ell_s(P)}{\ell_r(P)} \leq \frac{\ell(t)}{\ell_r(P)} = \frac{\ell(t')+md_x(U_1,V_1)}{\ell_r(P')+d_x(U_1,V_1)} < \rho_n ,$$

a contradiction to the fact that $\rho(P) = \rho_n$.

Corollary: Suppose $\rho_n = k$ where k is a positive integer. Then there exists a set P of n points with $\rho(P) = \rho_n$ such that each side of R(P) contains at least $k+1$ points of P.

Proof: We follow the notation in Lemma 3. Let

$$n' = \mathrm{Min}\{x:\rho_x=k\}.$$

Suppose line AB contains m points, $m \leq k$. Let CDEF be the smallest rectangle enclosing all other points of P not on line AB. Then line EF contains a point of P. But no V_i, $i = 1,\ldots,m$, can be a point of P since otherwise $|P'| < n'$ implies $k > \rho(P') \geq \frac{\ell(t)}{\ell_r(P)} \geq k$, an absurdity. Therefore line EF contains at least $m+1$ points of P'. So P' is a set of n' points with $\rho(P') = \rho_{n'}$ and having at least one more point on a side which previously contained less than $k+1$ points. Repeating this construction, we obtain in a finite number of steps a set P* of n' points with $\rho(P^*) = \rho_{n'}$, and each side of R(P*) containing at least $k+1$ points of P*.

Suppose $n > n'$. Let P** be the union of P* and any $n - n'$ points inside R(P*). Then $\rho(P^{**}) = \rho_n$ and each side of R(P**) contains at least $k+1$ points of P**.

Lemma 4: There exists a set P of n points with $\rho(P) = \rho_n$, $n \geq 4$, such that all four points at the corners of R(P) are points of P.

Proof: Let R(P) be the rectangle ABCD as shown in Figure 2. Assume $A \notin P$.

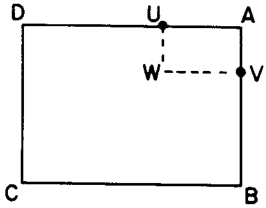

Fig. 2

From Lemma 3, we may assume that P is such that line AB and line AD each contain at least two points. Let U be the rightmost point on line AD and V the uppermost point on line AB. Consider the rectangle AVWU. Let Q be the set of points of P lying in the rectangle AVWU.

Let X be a point in $Q \cup \{V\} - \{U\}$ such that no other point in Q is higher than X, and X is the rightmost point among all points in Q which has the same height as X. Draw the rectangle UU'XX' as shown in Figure 3.

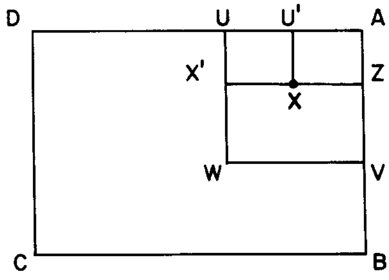

Fig. 3

Define $P' = P \cup \{U'\} \cup \{X'\} - \{U\} - \{X\}$.

It is easy to see that $\ell_r(P') = \ell_r(P)$. On the other hand we will show $\ell_s(P') \geq \ell_s(P)$. This implies $\rho(P') = \rho_n$. Thus, in a finite number of steps, we will find a set P^* with $\rho(P^*) = \rho_n$ and $\{A,B,C,D\} \subseteq P^*$.

Consider a minimal RST t' for P'. We may assume the intersection of t' and the open rectangle AZXU' is of length zero according to the dimension reduction theorem in [6]. Suppose the intersection q of t' and the closed rectangle UU'XX' is a forest of length not less than the sum of d(X,X') and d(U,X'). Let t be the union of t' - q with line XX' and UX'. Then

$$\ell_s(P') \geq \ell(t) \geq \ell_s(P).$$

Therefore we may assume that q is of length less than the sum of d(X,X') and d(U,X'). Let q' be a connected component of q which contains U'. Then clearly q' does not contain X', for otherwise q' would be longer than q. Let G be a point on line XX' or line UX' and suppose G is in q'. Without loss of generality we may assume G is on line XX'. Then

$$\ell(q) \geq d(U'X) + d(G,X).$$

Let t be the union of t' - q and lines UX',GX. Then t is a Steiner tree for P. Hence

$$\ell_s(P') \geq \ell(t) \geq \ell_s(P),$$

and Lemma 4 is proven.

3. ρ_n FOR $6 \leq n \leq 8$

Theorem 2: $\rho_6 = 5/3$.

Proof: First, we have $\rho_6 \geq 5/3$ by letting $a = 3$ and $b = 2$ in Lemma 2.

Let $P = \{A,B,C,D,X,Y\}$ be a set of points with $\rho(P) = \rho_6$. From Lemma 4, we may assume that $R(P)$ is the rectangle ABCD as shown in Figure 4. Label the other two points from left to right by X and Y.

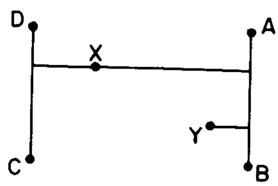

Fig. 4

Let t be an RST for P as shown in Figure 4. We have

$$\ell(t) \leq 2d(A,B) + d(A,D) + d_x(A,Y).$$

Similarly we have

$$\ell(t) \leq 2d(A,B) + d(A,D) + d_x(C,X).$$

Let t' be an RST for P as constructed in Figure 5.

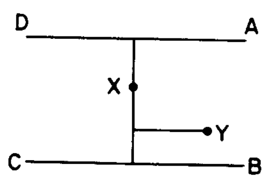

Fig. 5

We have

$$\ell(t') \leq d(A,B) + 2d(A,D) + d_x(X,Y).$$

Therefore

$$
\begin{aligned}
3\ell_s(P) &\le \ell(t) + \ell(t) + \ell(t') \\
&\le 5d(A,B) + 4d(A,D) + d_x(D,X) + d_x(X,Y) + d_x(Y,A) \\
&= 5(d(A,B) + d(A,D)) \\
&= 5\ell_r(P), \quad \text{or}
\end{aligned}
$$

$$\rho_n = \frac{\ell_s(P)}{\ell_r(P)} \le \frac{5}{3}.$$

This proves the theorem.

Theorem 3: $\rho_7 = 7/4$.

Proof: Let Q be the set of points as shown in Figure 6. Then we have $\rho_7 \ge \rho(Q) = 7/4$.

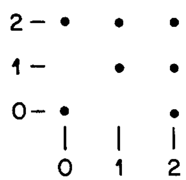

Fig. 6

Let P = {A,B,C,D,X,Y,Z} be a set of points with $\rho(P) = \rho_7$. From Lemma 4, we may assume that R(P) is the rectangle ABCD as shown in Figure 7. Label the other three points from left to right by X,Y and Z.

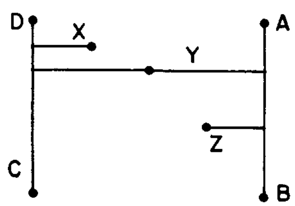

Fig. 7

Let t be an RST for P as constructed in Figure 7. We have

$$\ell(t) \leq 2d(A,B) + d(A,D) + d_x(D,X) + d_x(Z,A).$$

Let t' be an RST for P as constructed in Figure 8. We have

$$\ell(t') \leq 2d(A,D) + d(A,B) + d_x(X,Z).$$

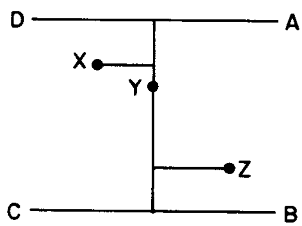

Fig. 8

Therefore

$$\begin{aligned} 2\ell_s(P) &\leq \ell(t) + \ell(t') \\ &\leq 3d(A,B) + 3d(A,D) + d_x(D,X) + d_x(X,Z) + d_x(Z,A) \\ &= 3d(A,B) + 4d(A,D). \end{aligned}$$

Similarly we can show

$$2\ell_s(P) \leq 4d(A,B) + 3d(A,D).$$

It follows that

$$4\ell_s(P) \leq 7\ell_r(P).$$

The theorem is proven.

Theorem 4: $\rho_8 = 11/6$.

Proof: Let Q be the set of points as shown in Figure 9. Then we have $\rho_8 \geq \rho(Q) = 11/6$.

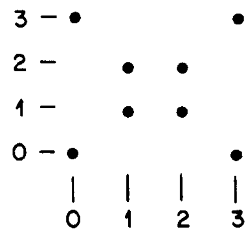

Fig. 9

Let $P = \{A,B,C,D,E,F,G,H\}$ be a set of points with $\rho(P) = \rho_8$. From Lemma 4, we may assume that $R(P)$ is the rectangle ABCD as shown in Figure 10. Label the other four points from left to right by E,F,G and H.

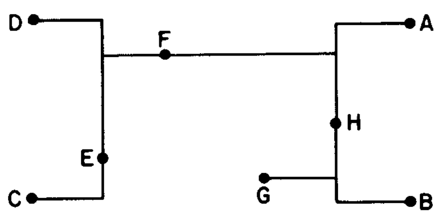

Fig. 10

Let t be the RST for P as constructed in Figure 10. Then

$$\ell(t) = 2\ell_r(P) - d_x(E,G).$$

Let t' be the RST for P as constructed in Figure 11.

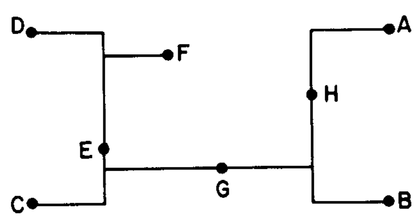

Fig. 11

Then

$$\ell(t') = 2\ell_r(P) - d_x(F,H).$$

Let t" be the RST for P as constructed in Figure 12.

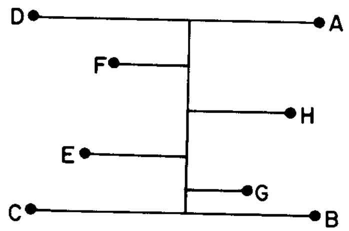

Fig. 12

Then

$$\ell(t") = \ell_r(P) + d(A,D) + d_x(E,G) + d_x(F,H).$$

Therefore,

$$3\ell_s(P) \le \ell(t) + \ell(t') + \ell(t")$$
$$\le 5\ell_r(P) + d(A,D).$$

Similarly we can show

$$3\ell_s(P) \le 5\ell_r(P) + d(A,B).$$

It follows that

$$6\ell_s(P) \le 11\ell_r(P).$$

The theorem is proven.

4. ρ_9 AND ρ_{10}

Theorem 5: $\rho_9 = \rho_{10} = 2$.

Proof: By setting a = b = 3 in Lemma 2, we obtain $\rho_{10} \ge \rho_9 \ge 2$. Therefore, it suffices to prove $\rho_{10} \le 2$.

Let P = {A,B,C,D,E,F,G,H,X,Y} be a set of points with $\rho(P) = \rho_{10}$. By the Corollary of Lemma 3, we may assume that R(P) is the rectangle ABCD and E,F,G,H are on the four sides of R(P) respectively as shown in Figure 13. Label the other two points by X and Y with X being the lower point.

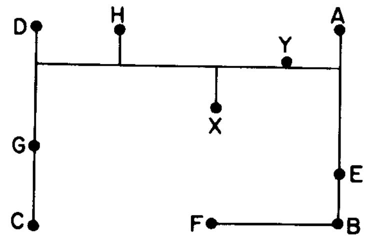

Fig. 13

Without loss of generality, we assume $d(A,B) \leq d(A,D)$ and Y is in the first quadrant.

Case (i): X is either in the first or the second quadrant.

Construct t as in Figure 13. The only requirement is that F should be connected to either B or C depending on which point is closer. We have

$$\begin{aligned} \ell(t) &= d(A,D) + 2d(A,B) + d_x(F,B) + d_y(H,X) \\ &\leq d(A,D) + 2d(A,B) + \tfrac{1}{2}d(A,D) + \tfrac{1}{2}d(A,B) \\ &\leq 2\ell_r(P). \end{aligned}$$

Case (ii): X is in the fourth quadrant (see Figure 14). Without loss of generality, assume $d(A,E) \leq d(E,B)$.

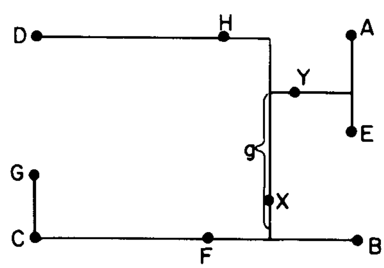

Fig. 14

(a) H is to the left of Y and X.

Construct t as in Figure 14 where G is connected to C or D depending on which is closer. If line AE is too short to meet the horizontal line from Y, then extend it to meet. Clearly we have $\ell(t) \leq 2\ell_r(P)$. If Y is to the left of X, then the line segment g should be shifted to the right.

(b) X is to the left of Y and H.

Construct t either as in Figure 15 or as in Figure 16 depending on whether H is to the left of Y or not.

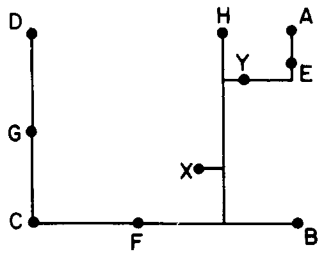

Fig. 15

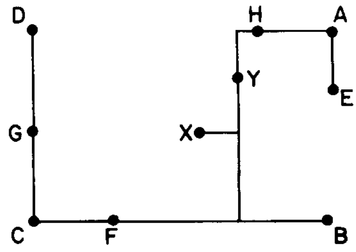

Fig. 16

We have

$$\ell(t) = d(A,D) + 2d(A,B) + d_x(E,X) + \text{Max}\{d_y(A,E), d_y(A,Y)\}$$

$$\leq d(A,D) + 2d(A,B) + \frac{1}{2}d(A,D) + \frac{1}{2}d(A,B)$$

$$\leq 2\ell_r(P).$$

(c) Y is to the left of X and H.

Construct t as in Figure 17. (If H is to the right of X, extend AH to meet the vertical line through X.)

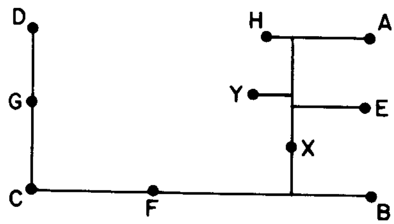

Fig. 17

We have

$$\ell(t) = d(A,D) + 2d(A,B) + d_x(E,Y) + \text{Max}\{d_x(A,H), d_x(A,X)\}$$

$$\leq d(A,D) + 2d(A,B) + \frac{1}{2}d(A,D) + \frac{1}{2}d(A,D)$$

$$= 2\ell_r(P).$$

Case (iii): X is in the third quadrant.

Suppose F and H are in the same vertical half in R(P), say on the left half. Then construct t as in Figure 18.

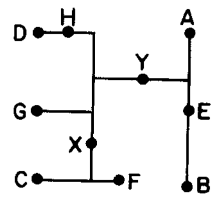

Fig. 18

Clearly

$$\ell(t) \leq 2\ell_r(P).$$

Therefore from now on, we assume F and H (E and G) are in different halves of R(P). We also assume without loss of generality that

$$d(A,H) + d(B,F) \leq d(H,D) + d(C,F).$$

There are five subcases to be studied.

(a) Y is to the right of H and F.

Construct t as in Figure 19.

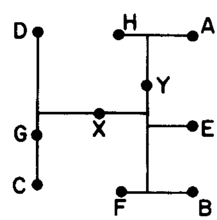

Fig. 19

$$\ell(t) = 2d(A,B) + d(A,D) + d_x(A,H) + d_x(B,F) \leq 2\ell_r(P).$$

(b) H is to the right of Y.

Construct t as in Figure 20 if F is to the left of X.

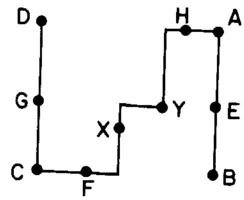

Fig. 20

$$\ell(t) = 3d(A,B) + d(A,D) \leq 2\ell_r(P).$$

Construct t and t' as in Figures 21 and 22 if F is to the right of X and to the left of Y.

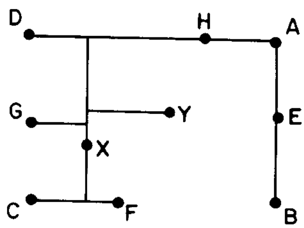

Fig. 21

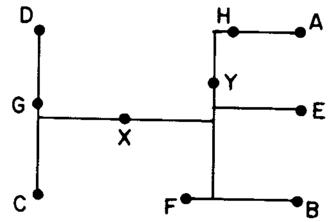

Fig. 22

$$2\ell_s(P) \leq \ell(t) + \ell(t') = 4\ell_r(P).$$

(c) F is to the right of Y. E is above Y.

Construct t as in Figure 23.

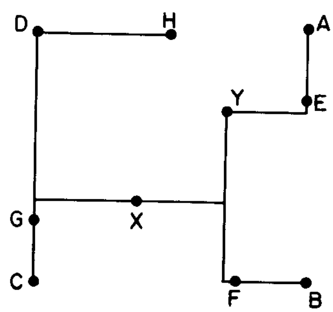

Fig. 23

$$\ell(t) = 2d(A,B) + d(A,D) + d_x(A,Y) + d_x(H,D)$$

$$= 2d(A,B) + d(A,D) + \frac{1}{2}d(A,D) + \frac{1}{2}d(A,D)$$

$$= 2\ell_r(P).$$

(d) F is to the right of Y. E is below Y. G is above X. Construct t and t' as in Figures 24 and 25.

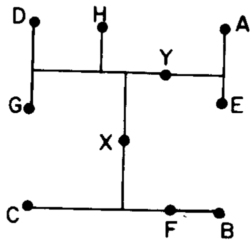

Fig. 24

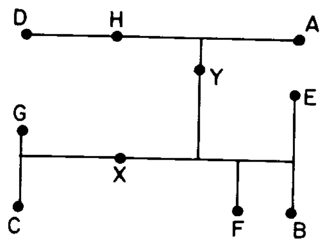

Fig. 25

$$2\ell_s(P) \leq \ell(t) + \ell(t') = 4\ell_r(P).$$

(e) F is to the right of Y. E is below Y. G is below X. Construct t as in Figure 26.

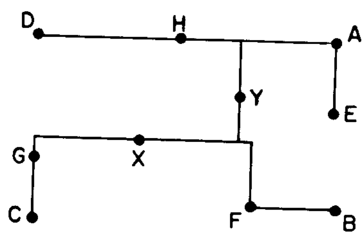

Fig. 26

$$\begin{aligned}\ell(t) &= 2d(A,D) + d(A,B) + d(A,E) + d_y(C,X) \\ &\leq 2d(A,D) + d(A,B) + \tfrac{1}{2}d(A,B) + \tfrac{1}{2}d(A,B) \\ &= 2\ell_r(P).\end{aligned}$$

Thus all cases have been covered and the theorem is proven.

We have not been able to determine ρ_{11} except that the set Q of eleven points in Figure 27 shows $\rho_{11} \geq \rho(Q) = \frac{33}{16}$.

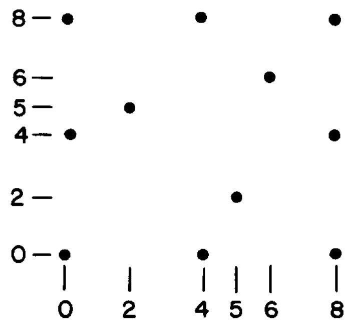

Fig. 27

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