



Multidiameters and Multiplicities

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The k -diameter of a graph Γ is the largest pairwise minimum distance of a set of k vertices in Γ , i.e., the best possible distance of a code of size k in Γ . A k -diameter for some k is called a multidiameter of the graph. We study the function $N(k, \Delta, D)$, the largest size of a graph of degree at most Δ and k -diameter D . The graphical analogues of the Gilbert bound and the Hamming bound in coding theory are derived. Constructions of large graphs with given degree and k -diameter are given. Eigenvalue upper bounds are obtained. By combining sphere packing arguments and eigenvalue bounds, new lower bounds on spectral multiplicity are derived. A bound on the error coefficient of a binary code is given.

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1. INTRODUCTION

The diameter of a graph measures how far two distinct points can be; similarly, the k -diameter (definition given below) measures how far k points can be; in other words, how good can a code of size k in the graph be. Two problems on the diameter have excited a great deal of attention since the 1980s:

- the (Δ, D) graph problem: how large can a graph of bounded degree and given diameter be? [1, 6, 7]
- finding the best spectral upper bound on the diameter of a graph [3, 4].

This work is an attempt to generalize both philosophies.

First, we study a function $N(k, \Delta, D)$, the largest size of a graph of degree at the most Δ and given k -diameter D . Observe that this is a very hard problem which comprises as a special case ($k = 2$) the (Δ, D) graph problem [1, 6, 7]. We begin a tridimensional table collecting the sizes of the largest such graphs. No exact value of $N(k, \Delta, D)$ with $k > 2$, $\Delta > 2$ is known so far.

Next, we derive the natural analogues of the Chung *et al.* upper bounds [3, 4, 8] on the diameter. Combining the spectral bound and the Gilbert bound, we obtain some new lower bounds on the spectral multiplicity. When the graph under scrutiny is the coset graph of a binary linear code, we obtain a lower bound on the error coefficient of the dual code.

2. DEFINITIONS AND NOTATIONS

All graphs Γ considered are finite, connected, with vertex set V of cardinality v , simple, undirected, without loops or multiple edges. The graphical *distance* $d(x, y)$ between two vertices x and y is the length of a shortest path between x and y . The *girth* hereby denoted by g is the shortest size of a circuit. A k -code in such a graph Γ with *distance* d is a set of $k (\geq 2)$ vertices with

$$\min_{i \neq j} (d(x_i, x_j)) = d.$$

Classical codes of size k in the sense of [10] are exactly k -codes in the hypercube or more generally the q -ary hypercube or Hamming graph [2] $H(n, q)$. The k -diameter of Γ , say D_k , is the largest possible distance a k -code in Γ can have. Note that D_2 is the standard diameter.

For example, we have $D_q = n$ in the q -ary Hamming graph [2] $H(n, q)$ (the direct sum of n complete graphs K_q). The sequence $k \mapsto D_k$ is nonincreasing:

$$D_2 \geq D_3 \geq \cdots D_k \geq D_{k+1} \geq \cdots.$$

A closely related and useful quantity is the *Moore function*, which is defined for integers Δ, D

$$M(\Delta, D) := 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1},$$

or, in closed form

$$M(\Delta, D) = \begin{cases} \frac{\Delta(\Delta-1)^{D-2}}{\Delta-2} & \text{if } \Delta > 2, \\ 2D + 1 & \text{if } \Delta = 2. \end{cases}$$

An elementary fact of graph theory is that a graph of diameter at most D and degree at most Δ cannot have more than $M(\Delta, D)$ vertices. Its bipartite counterpart $M_b(\Delta, D)$ is $2(1 + (\Delta - 1) + \cdots + (\Delta - 1)^{D-1})$, or, in closed form

$$M_b(\Delta, D) = \begin{cases} \frac{2(\Delta-1)^{D-2}}{\Delta-2} & \text{if } \Delta > 2, \\ 2D & \text{if } \Delta = 2. \end{cases}$$

A similar but different invariant is the *covering radius* $r(C)$ of a code C which is defined by

$$r(C) := \max_{v \in V} \min_{c \in C} d(v, c).$$

Denote by T the diagonal matrix indexed by V such that $T_{x,x}$ is the degree of $x \in V$, by A the adjacency matrix of Γ and let $L = T - A$. The *Laplace operator* is then defined as

$$\mathcal{L} := T^{-1/2} L T^{-1/2}.$$

Let

$$\lambda_0 = 0 \leq \lambda_1 \leq \cdots \leq \lambda_{v-1}$$

be the eigenvalues of the Laplace operator arranged in increasing order. It is not hard to check that the whole *spectrum* fits into $[0, 2]$. In particular, if the graph is Δ -regular, then $\lambda_i = 1 - \mu_i/\Delta$, where μ_i is the i th eigenvalue (decreasing order) of the adjacency matrix.

3. SPHERE PACKINGS AND COVERINGS

3.1. An improved Gilbert bound. The Moore function is an upper bound for the number of vertices in a graph with degree Δ and diameter D . Using the definition of the 2-diameter, the Moore bound is related to $N(2, \Delta, D)$ as follows:

$$N(2, \Delta, D) \leq M(\Delta, D). \quad (1)$$

The analogue of the Gilbert bound of coding theory [10, p. 33] is

$$N(k, \Delta, D) \leq (k - 1)M(\Delta, D),$$

which can be proved by induction on k (equation (1) is the case $k = 2$). It also follows immediately from the following result relating $N(k, \Delta, D)$ to codes with covering radius D .

LEMMA 1.

$$N(k, \Delta, D) \leq F(k, \Delta, D), \quad (2)$$

where $F(k, \Delta, D)$ denotes the largest size of the vertex set V , of a graph Γ containing a $(k - 1)$ -code C with minimum distance at least $D + 1$, and covering radius of at most D .

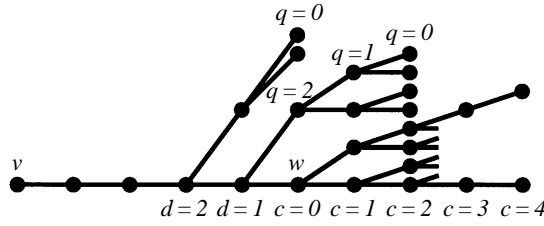


FIGURE 1. Explanation of $G(\Delta, D)$.

PROOF. Let Γ be a graph with degree at most Δ , satisfying $D_k \leq D$ on $N(k, \Delta, D)$ vertices. Consider a $(k - 1)$ -code C with minimum distance at least $D + 1$ in Γ . Then we conclude that its covering radius is at most D , since otherwise, we would obtain a k -code with distance $D + 1$, contradicting the hypothesis $D_k \leq D$. \square

We are now in a position to give a small improvement on the Gilbert bound by taking into account graph connectedness and ball intersections.

THEOREM 1. *A graph with k -diameter D and maximum degree Δ has at most $(k - 1)M(\Delta, D) - (k - 2)M(\Delta, (D - 1)/2)$ vertices if D is odd and $(k - 1)M(\Delta, D) - (k - 2)M_b(\Delta, D/2)$ if D is even.*

PROOF. If the graph is connected, we can see that if v and w are vertices at a distance $D + 1$, then the number of vertices at a distance at most D from w and at least $D + 1$ from v is at most,

$$I(\Delta, D) := \sum_{c=0}^D (\Delta - 1)^c + \sum_{d=1}^{\lfloor D/2 \rfloor} (\Delta - 2) \sum_{q=0}^{D-2d} (\Delta - 1)^{D-d-1-q}.$$

The explanation of that formula should be clear from Figure 1. We note that $I(\Delta, D) = M(\Delta, D) - M(\Delta, (D - 1)/2)$ if D is odd and $I(\Delta, D) = M(\Delta, D) - M_b(\Delta, D/2)$ if D is even. These formulae can be derived by summing the geometric series in the definition of $G(\Delta, D)$. Roughly speaking, what happens for g large enough, and say D is odd, is that the intersection of the two balls of radius D about v and w , is itself a ball of radius $\lfloor (D - 1)/2 \rfloor$ centered in a point at the same distance (up to one unit) of v and w . A similar phenomenon occurs for even D .

To apply Lemma 1, we construct greedily a code of minimum distance $> D$ and covering radius at most D . The first ball covers at the most, $M(\Delta, D)$ vertices. Put the second one at distance $D + 1$ from the first. It covers at the most $M(\Delta, D) - G(\Delta, D)$ vertices. After placing the $(k - 1)$ th ball, at most $(k - 2)M(\Delta, D) - G(\Delta, D)$ vertices are covered. \square

For $\Delta = 2$, the bound is $2D + 1 + (k - 2)(D + 1)$, instead of $(k - 1)(2D + 1)$, the exact value is $k(D + 1) - 1$ (and the optimal graph is a cycle).

3.2. *The Hamming bound.* Let $e(D) := \lfloor \frac{D-1}{2} \rfloor$. Assume $D_k \leq g$. Then the analogue of the Hamming bound [10] reads

$$v \geq kM(\Delta, e(D_k)).$$

Equality corresponds to the case of a *perfect code* in a graph such that the volume of a ball of radius $e(D)$ is indeed $M(\Delta, e(D))$. This happens, for instance, if the graph is regular with girth at least $2e(D) + 1$. For instance, if $n = 2^m - 1$, the Hamming codes yield

$$N(2^{n-m}, n, 3) = 2^n = 2^{n-m}(2n + 1),$$

meeting the preceding bound with equality.

Similarly perfect Lee metric codes of length n over F_p with $p = 2n + 1$ yield

$$N(p^{n-1}, 2n, 3) = p^n = p^{n-1}(1 + 2n).$$

But the repetition codes of length $2s + 1$ yield only (for $s > 1$) the inequality

$$N(2, 2s + 1, 2s + 1) \geq 2^{2s+1}.$$

Observe that the volume of a ball of radius s in the $(2s + 1)$ -hypercube is $2^{2s} < M(2s + 1, 2s + 1)$.

4. CONSTRUCTIONS

4.1. *Graphs of large girth.* A relation between the girth and the k -diameter is

$$g \leq kD_k + 1.$$

This bound is of special interest in the case of incidence graphs of generalized polygons [7]. These are bipartite graphs with diameter N and girth $2N$. In that case, we obtain the estimates

$$\left\lceil \frac{2N - 1}{k} \right\rceil \leq D_k \leq N.$$

The lower bound is met with equality for the $N = 3, k = 3$ case of the incidence graph of a projective plane $PG(2, q)$. It can be directly verified in that case that a line l , along with a pair of points known on l , constitutes a 3-code with distance 2, and also that $D_3 \leq 2$, because a pair of points or a pair of lines are at a distance at most 2 apart. So, we obtain, for every prime power q , the estimates

$$N(3, q + 1, 2) \geq 2(q^2 + q + 1),$$

quite close to the upper bound

$$N(3, q + 1, 2) \leq 2(2 + 2q + q^2) - 2.$$

Similarly, in that case, we have $D_4 = 2$ yielding

$$N(4, q + 1, 2) \geq 2(q^2 + q + 1),$$

to be compared to the upper bound

$$N(3, q + 1, 2) \leq 3(2 + 2q + q^2) - 4.$$

A sharper lower bound will appear in Section 4.3.

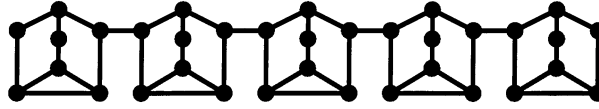


FIGURE 2. Graph showing $N(k, 3, 3) \geq 7(k - 1)$.

4.2. *Bipartite Graphs.* Let $B(\Delta, D)$ denote the largest size of a bipartite graph of degree at most Δ and diameter D . Lower bounds for this function are tabulated for $\Delta \leq 16$ and $D \leq 10$ in [6], and an upper bound is $M_b(\Delta, D)$. Given a 3-code in such a graph, two of its points at least should lie in the same part. This leads to

$$N(3, \Delta, D - 1) \geq B(\Delta, D) \quad \text{if } D \text{ is odd.}$$

This can be generalized to multipartite graphs by denoting by $P(p, \Delta, D)$ the largest size of a p -partite graph with diameter D and degree at most Δ . For any $D = pm + a$ with $0 < a < p$ and m integer, we have

$$N(p + 1, \Delta, pm) \geq P(p, \Delta, D).$$

4.3. *Irregular Graphs.* For $D = 2$ and $\Delta = q + 1$, we have some constructions with $n = (q^2 + q + 1)(k - 1) = (\Delta^2 - \Delta + 1)(k - 1)$ (take $k - 1$ copies of the quotient of the incidence graph of a projective plane of order q by a polarity and add edges between the vertices of degree q to obtain a connected graph). On the other hand, the modified Gilbert bound is $(\Delta^2 - 1)(k - 1) + 2$ instead of $(\Delta^2 + 1)(k - 1)$.

For $q = 2$, we have the kind of graphs of Figure 2.

More generally, if we have a graph with maximum degree $\leq \Delta$ and diameter D , with n vertices among which two at least have degree $\Delta - 1$, it is easy to prove $N(k, \Delta, D) \geq (k - 1)n$ by arranging a path of $k - 1$ copies of G connected by edges.

This gives $N(k, \Delta, 1) \geq (k - 1)\Delta$, to be compared by the upper bound coming from $k - 1 \geq \lceil \frac{n}{\Delta} \rceil$, namely $N(k, \Delta, 1) \leq 1 + (k - 1)\Delta$.

For $D = 3, 5$, the generalized quadrangles and hexagons also give good results: a (s, t) -quadrangle Q has $(s + 1)(st + 1)$ vertices of degree $t + 1$ and $(t + 1)(st + 1)$ vertices of degree $s + 1$; one of the components of the Kronecker product of Q by itself is regular of degree $(t + 1)(s + 1)$, has $2(s + 1)(t + 1)(st + 1)^2$ vertices and admits an obvious polarity, the quotient is a diameter 3 graph with $(s + 1)(t + 1)(st + 1)^2$ vertices of maximum degree $\Delta = (t + 1)(s + 1)$, with $(s + 1)(t + 1)(st + 1)$ vertices having degree $st + t + s$. Hence $N(k, (t + 1)(s + 1), 3) \geq (k - 1)(s + 1)(t + 1)(st + 1)^2$, provided that there exists a (s, t) -quadrangle.

For each $m \geq 0$, some quadrangles with $s = t = 2^{2m+1}$ already have a polarity and give $N(k, s + 1, 3) \geq (k - 1)(s + 1)(s^2 + 1)$.

For example, $15(k - 1) \leq N(k, 3, 3) \leq 18(k - 1) + 4$.

Similarly, the (s, t) -hexagons yield graphs with $(s + 1)(t + 1)(s^2t^2 + st + 1)^2$ vertices, maximum degree $\Delta = (t + 1)(s + 1)$, with $(s + 1)(t + 1)(s^2t^2 + st + 1)$ vertices having degree $st + t + s$. Hence, $N(k, (t + 1)(s + 1), 5) \geq (k - 1)(s + 1)(t + 1)(s^2t^2 + st + 1)^2$, provided that there exists a (s, t) -generalized hexagon, and hexagons of order $s = t = 3^{2m+1}$ admitting a polarity give $N(k, s + 1, 5) \geq (k - 1)(s + 1)(s^2 + s + 1)$.

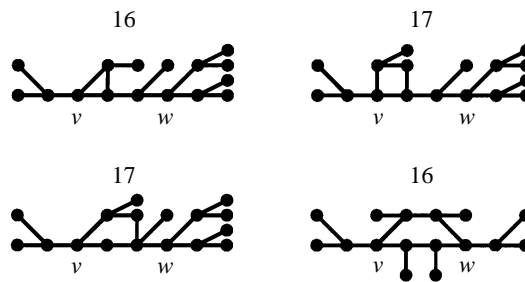


FIGURE 3. Explanation of $N(3, 3, 2) \leq 16$.

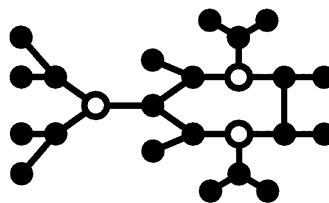


FIGURE 4. Explanation of $N(4, 3, 2) \leq 24$.

4.4. *Small values* For $D = 2, k = 3$ and $\Delta = 3$, we have $M = 10$ and $F = 8$; the improved Gilbert bound is 18.

If there is a vertex v of degree ≤ 2 , there are at the most seven vertices at a distance ≤ 2 from v . If w is at a distance 3 from v , there are at the most eight vertices at a distance ≤ 2 from w and at a distance at least 3 from v . Thus, a graph with maximum degree 3 and 3-diameter 2 having a vertex of degree 2 has at the most 15 vertices. Thus, there is no graph with 17 vertices, maximum degree 3 and 3-diameter 2.

Let us consider now 3-regular connected graphs. Since $18 < M(3, 3) = 22$, every vertex of a 3-regular connected graph with 3-diameter 2 lies in a cycle of length < 7 , thus the bound 18 cannot be attained. See Figure 3.

Thus we can state $14 \leq N(3, 3, 2) \leq 16$.

Similarly, from the improved Gilbert bound for $N(4, 3, 2)$, that is 26, we can see that the girth is at the most 7, and this implies $N(4, 3, 2) \leq 24$ because we can either find a vertex of degree ≤ 2 leading to a bound of $7 + 8 + 8 = 23$ or use the results for a cycle of length ≤ 6 , or start with a cycle of length 7 as shown in Figure 4. The proof is easily generalized to $N(k, 3, 2) \leq 8(k - 1)$.

TABLE 1.

Small values.						
k	Δ	D	$N(k, \Delta, D)$	G	IG	graph
3	3	2	14–16	20	18	$PG(2, 2)$
3	3	4	56–68	72	68	[6]
3	3	3	30–40	66	40	4.3
3	4	2	26–32	34	32	$PG(2, 3)$
4	3	2	21–24	30	26	Section 4.3
4	3	3	45–58	66	58	Section 4.3

The fourth column gives the known bounds, the fifth and sixth ones present the Gilbert and

improved Gilbert bounds.

5. SPECTRAL BOUNDS

5.1. *Main bounds.* Recall that the inverse hyperbolic cosine is given by

$$\cosh^{-1}(x) = \log\left(x + \sqrt{x^2 - 1}\right),$$

for $x \geq 1$.

THEOREM 2. *Suppose a connected graph G is not a complete graph. For $X, Y \subset V(G)$ and X not equal to the complement \bar{Y} of Y , we have*

$$d(X, Y) \leq \left\lceil \frac{\cosh^{-1} \sqrt{\frac{\text{vol } X \text{ vol } \bar{Y}}{\text{vol } X \text{ vol } Y}}}{\cosh^{-1} \frac{\lambda_{v-1} + \lambda_1}{\lambda_{v-1} - \lambda_1}} \right\rceil. \tag{3}$$

PROOF. For $X \subset V(G)$, we define

$$\psi_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

If we can show that for some integer t and some polynomial $p_t(z)$ of degree t ,

$$\langle T^{1/2} \psi_Y, p_t(\mathcal{L})(T^{1/2} \psi_X) \rangle > 0,$$

then, there is a path of length at the most t joining a vertex in X to a vertex in Y . Therefore, we have $d(X, Y) \leq t$.

Let a_i denote the Fourier coefficients of $T^{1/2} \psi_X$, i.e.,

$$T^{1/2} \psi_X = \sum_{i=0}^{n-1} a_i \phi_i,$$

where the ϕ_i 's are the orthonormal eigenfunctions of \mathcal{L} . In particular, we have

$$a_0 = \frac{\langle T^{1/2} \psi_X, T^{1/2} \mathbf{1} \rangle}{\langle T^{1/2} \mathbf{1}, T^{1/2} \mathbf{1} \rangle} = \frac{\text{vol } X}{\text{vol } G},$$

where $\text{vol } X$ is the sum of degrees of the vertices in X and $\mathbf{1}$ denotes all the 1's functions.

Similarly, we write

$$T^{1/2} \psi_Y = \sum_{i=0}^{v-1} b_i \phi_i,$$

Suppose we choose $p_1(z) = \frac{2z}{\lambda_{v-1} + \lambda_1} - 1$ and $p_t(z) = (p_1(z))^t$. Since G is not a complete graph, $\lambda_1 \neq \lambda_{v-1}$, and

$$|p_t(\lambda_i)| \leq (1 - \lambda)^t$$

for all $i = 1 \dots, v - 1$, where $\lambda = 2\lambda_1 / (\lambda_{v-1} + \lambda_1)$. Therefore, we have

$$\begin{aligned} \langle T^{1/2} \psi_Y, p_t(\mathcal{L})T^{1/2} \psi_X \rangle &= a_0 b_0 + \sum_{i>0} p_t(\lambda_i) a_i b_i \\ &\geq a_0 b_0 - (1 - \lambda)^t \sqrt{\sum_{i>0} a_i^2 \sum_{i>0} b_i^2} \\ &= \frac{\text{vol } X \text{ vol } Y}{\text{vol } G} - (1 - \lambda)^t \frac{\sqrt{\text{vol } X \text{ vol } \bar{X} \text{ vol } Y \text{ vol } \bar{Y}}}{\text{vol } G}, \end{aligned}$$

by using the fact that

$$\begin{aligned} \sum_{i>0} a_i^2 &= \|T^{1/2}\psi_X\|^2 - \frac{(\text{vol } X)^2}{\text{vol } G} \\ &= \frac{\text{vol } X \text{ vol } \bar{X}}{\text{vol } G}. \end{aligned}$$

We note that in the above inequality, the equality holds if and only if $a_i = cb_i$ for $i > 0$, for some constant c . This can only hold when $X = Y$ or $X = \bar{Y}$. Since the theorem obviously holds for $X = Y$ and we have the hypothesis that $X \neq \bar{Y}$, we may assume that the inequality is strict. If we choose

$$t \geq \frac{\log \sqrt{\frac{\text{vol } \bar{X} \text{ vol } \bar{Y}}{\text{vol } X \text{ vol } Y}}}{\log \frac{1}{1-\lambda}},$$

we have

$$\langle T^{1/2}\psi_Y, p_t(\mathcal{L})T^{1/2}\psi_X \rangle > 0,$$

This proves $D(G) \leq \left\lceil \frac{\log \sqrt{\frac{\text{vol } \bar{X} \text{ vol } \bar{Y}}{\text{vol } X \text{ vol } Y}}}{\log \frac{1}{1-\lambda}} \right\rceil$.

It can be improved with another choice for p_t , namely the normalized Chebychev polynomial such that

$$\cosh\left(t \cosh^{-1}\left(\frac{1}{1-\lambda}\right)\right) p_t(z) := \cosh\left(t \cosh^{-1}\left(\frac{1}{1-\lambda} - \frac{2z}{\lambda_{v-1} - \lambda_1}\right)\right).$$

Then, for $\lambda_1 \leq z \leq \lambda_{v-1}$, we have $|p_t(z)| \leq \frac{1}{\cosh\left(t \cosh^{-1}\left(\frac{1}{1-\lambda}\right)\right)}$,

$$\begin{aligned} \langle T^{1/2}\psi_Y, p_t(\mathcal{L})T^{1/2}\psi_X \rangle &= a_0b_0 + \sum_{i>0} p_t(\lambda_i)a_i b_i \\ &\geq a_0b_0 - \frac{1}{\cosh\left(t \cosh^{-1}\left(\frac{1}{1-\lambda}\right)\right)} \sqrt{\sum_{i>0} a_i^2 \sum_{i>0} b_i^2} \\ &= \frac{\text{vol } X \text{ vol } Y}{\text{vol } G} - \frac{1}{\cosh\left(t \cosh^{-1}\left(\frac{1}{1-\lambda}\right)\right)} \frac{\sqrt{\text{vol } X \text{ vol } \bar{X} \text{ vol } Y \text{ vol } \bar{Y}}}{\text{vol } G}. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{i>0} a_i^2 &= \|T^{1/2}\psi_X\|^2 - \frac{(\text{vol } X)^2}{\text{vol } G} \\ &= \frac{\text{vol } X \text{ vol } \bar{X}}{\text{vol } G}. \end{aligned}$$

If we choose

$$t \geq \frac{\cosh^{-1} \sqrt{\frac{\text{vol } \bar{X} \text{ vol } \bar{Y}}{\text{vol } X \text{ vol } Y}}}{\cosh^{-1} \frac{1}{1-\lambda}},$$

we have

$$\langle T^{1/2}\psi_Y, p_t(\mathcal{L})T^{1/2}\psi_X \rangle > 0.$$

This proves Theorem 2

□

As an immediate consequence of Theorem 2, we have:

COROLLARY 1. Suppose G is a regular graph which is not complete. Then

$$D(G) \leq \left\lceil \frac{\cosh^{-1}(v-1)}{\cosh^{-1} \frac{\lambda_{v-1} + \lambda_1}{\lambda_{v-1} - \lambda_1}} \right\rceil.$$

To generalize Theorem 2 to distances among k subsets of the vertices, we need the following geometric lemma [5].

LEMMA 2. Let x_1, x_2, \dots, x_{d+2} denote $d+2$ arbitrary vectors in d -dimensional Euclidean space. Then there are two of them, say, $v_i, v_j, (i \neq j)$ such that $\langle v_i, v_j \rangle \geq 0$.

THEOREM 3. Suppose G is not a complete graph. For $X_i \subset V(G), i = 0, 1, \dots, k$, we have

$$\min_{i \neq j} d(X_i, X_j) \leq \max_{i \neq j} \left\lceil \frac{\cosh^{-1} \sqrt{\frac{\text{vol } \bar{X}_i \text{ vol } \bar{X}_j}{\text{vol } X_i \text{ vol } X_j}}}{\cosh^{-1} \frac{1}{1-\lambda_k}} \right\rceil,$$

if $1 - \lambda_k \geq \lambda_{v-1} - 1$.

PROOF. Let X and Y denote two distinct subsets among the X_i 's. We consider

$$\langle T^{1/2} \psi_Y, (I - \mathcal{L})^t T^{1/2} \psi_X \rangle \geq a_0 b_0 + \sum_{i=1}^{k-1} (1 - \lambda_i)^t a_i b_i - \sum_{i \geq k} (1 - \lambda_k)^t |a_i b_i|.$$

For each $X_i, i = 0, 1, \dots, k$, we consider the vector consisting of the Fourier coefficients of the eigenfunctions $\varphi_1, \dots, \varphi_{k-1}$ in the eigenfunction expansion of X_i . Suppose we define a scalar product for two such vectors (a_1, \dots, a_{k-1}) and (b_1, \dots, b_{k-1}) by

$$\sum_{i=1}^{k-1} (1 - \lambda_i)^t a_i b_i.$$

From Lemma 2, we know that we can choose two of the subsets, say, X and Y with their associated vectors satisfying

$$\sum_{i=1}^{k-1} (1 - \lambda_i)^t a_i b_i \geq 0.$$

Therefore, we have

$$\langle T^{1/2} \psi_Y, (I - \mathcal{L})^t T^{1/2} \psi_X \rangle > \frac{\text{vol } X \text{ vol } Y}{\text{vol } G} - (1 - \lambda_k)^t \frac{\sqrt{\text{vol } X \text{ vol } \bar{X} \text{ vol } Y \text{ vol } \bar{Y}}}{\text{vol } G}$$

and we proved that

$$\min_{i \neq j} d(X_i, X_j) \leq \max_{i \neq j} \left\lceil \frac{\log \sqrt{\frac{\text{vol } \bar{X}_i \text{ vol } \bar{X}_j}{\text{vol } X_i \text{ vol } X_j}}}{\log \frac{1}{1-\lambda_k}} \right\rceil.$$

By using the Chebychev polynomial $p_t(x)$ instead of $(1 - x)^t$, we can improve the above bound by replacing \log with \cosh^{-1} and Theorem 3 is proved. \square

We note that the condition $1 - \lambda_k \geq \lambda_{v-1} - 1$ can be eliminated by modifying the λ 's as follows:

THEOREM 4. For $X_i \subset V(G)$, $i = 0, 1, \dots, k$, we have

$$\min_{i \neq j} d(X_i, X_j) \leq \max_{i \neq j} \left[\frac{\cosh^{-1} \sqrt{\frac{\text{vol } \bar{X}_i \text{ vol } \bar{X}_j}{\text{vol } X_i \text{ vol } X_j}}}{\cosh^{-1} \frac{\lambda_{v-1} + \lambda_k}{\lambda_{v-1} - \lambda_k}} \right]$$

if $\lambda_k \neq \lambda_{v-1}$.

Observing that the denominator is a decreasing function of $\lambda_{v-1} \leq 2$, we obtain the following useful corollary.

COROLLARY 2. For $X_i \subset V(G)$, $i = 0, 1, \dots, k$, we have

$$\min_{i \neq j} d(X_i, X_j) \leq \max_{i \neq j} \left[\frac{\cosh^{-1} \sqrt{\frac{\text{vol } \bar{X}_i \text{ vol } \bar{X}_j}{\text{vol } X_i \text{ vol } X_j}}}{\cosh^{-1} \frac{2 + \lambda_k}{2 - \lambda_k}} \right]$$

if $\lambda_k \neq \lambda_{v-1}$.

Eventually, taking all X_i 's of size unity we obtain:

COROLLARY 3. If G is a graph of size v distinct from the complete graph its k -diameter is bounded above as

$$D_{k+1}(G) \leq \max_{i \neq j} \left[\frac{\cosh^{-1} v}{\cosh^{-1} \frac{2 + \lambda_k}{2 - \lambda_k}} \right]$$

if $\lambda_k \neq \lambda_{v-1}$.

5.2. *Spectral Multiplicity.* We now derive an application of the preceding bounds to estimate the spectral multiplicity of graphs.

THEOREM 5. If A_1 denotes the multiplicity of λ_1 for Δ regular graph on v vertices, then

$$A_1 + 1 \geq v/M \left(\Delta, \left[\frac{\cosh^{-1}(v)}{\cosh^{-1} \left(\frac{2 + \lambda_2}{2 - \lambda_2} \right)} \right] \right).$$

PROOF. Consider an $(A_1 + 1)$ -code of minimum distance D_{A_1+1} in the considered graph. Then, by Corollary 3, its minimum distance is at most $\lceil \cosh^{-1}(v) / \cosh^{-1}(\frac{2 + \lambda_2}{2 - \lambda_2}) \rceil$. We apply the analogue of the Gilbert bound mentioned before Lemma 1, keeping in mind that an upper bound on a ball of radius D in the graph we consider is $M(\Delta, D)$. \square

This can be generalized further to obtain bounds on the distribution of, say, the first m eigenvalues.

THEOREM 6. If A_i denotes the multiplicity of the i th distinct eigenvalue for a Δ regular graph on v vertices, and $k = \sum_{j=1}^m A_j$, then

$$\sum_{j=1}^m A_j + 1 \geq v/M \left(\Delta, \left[\frac{\cosh^{-1}(v)}{\cosh^{-1} \left(\frac{2 + \lambda_{k+1}}{2 - \lambda_{k+1}} \right)} \right] \right).$$

More generally, if we have a graph G with balls of radius D bounded by a function $M_G(D)$, we obtain the following result.

THEOREM 7. *With the above hypothesis, if A_i denotes the multiplicity of the i th distinct eigenvalue for a Δ regular graph on v vertices, and $k = \sum_{j=1}^m A_j$, then*

$$\sum_{j=1}^m A_j + 1 \geq v/M_G \left(\left[\frac{\cosh^{-1}(v)}{\cosh^{-1}\left(\frac{2+\lambda_{k+1}}{2-\lambda_{k+1}}\right)} \right] \right).$$

For instance for Abelian Cayley graphs of degree $\Delta = k_1 + 2k_2$ ($k_1 =$ number of generators of order 2), we obtain from [9]

$$M_G(D) = (2D + k)^k / k!,$$

where $k = k_1 + k_2$. Another graph that already received some attention in [8] is the *coset graph* of a linear code. Let C denote a k -dimensional binary linear code of length n , and minimum distance d . Then, a graph $G(C)$ can be built on the 2^k cosets of the dual code with degree n and spectrum related in a simple way to the weights w_i of C . Specifically, in the notations of Section 1, we have $\lambda_i = \frac{2w_i}{n}$, and $\lambda_1 = 2d/n$ with multiplicity A_d the so-called error coefficient in coding theory, which is the leading term in error probability calculations. It is straightforward to show that for this graph a suitable bound on the size of balls is

$$M_{G(C)}(D) = \sum_{j=0}^D \binom{n}{j},$$

or, using the entropy function $H(x) := -x \log_2(x) - (1 - x) \log_2(1 - x)$, and not necessarily performing asymptotics

$$M_{G(C)}(D) = 2^{nH(D/n)}.$$

Theorem 5 applied to that situation yields, noting that $G(C)$ is the complete graph if C is the dual of a perfect Hamming code.

THEOREM 8. *Let C be an $[n, k, d]$ binary linear code, which is not the dual of a perfect Hamming code and weights $w_1 = 0, w_1 = d, w_2, \dots, w_{n-1}$. Let*

$$U(n, k, x) := \left[\frac{\cosh^{-1}(2^k)}{\cosh^{-1}\left(\frac{n+x}{n-x}\right)} \right].$$

Then the error coefficient of C is bounded below as

$$A_d \geq 2^{k-nH(U(n,k,w_2)/n)}.$$

We leave it as an open problem to see if the exponent of the exponential on the RHS can be made > 0 .

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