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On Concentrators, Superconcentrators, Generalizers, and Nonblocking Networks

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In this paper, we study various communication networks, such as concentrators, superconcentrators, generalizers, and rearrangeable and nonblocking networks. We improve bounds for the number of edges (which can be viewed as approximations of the cost) in some networks by combinatorial analysis.

I. INTRODUCTION

A communication network can be viewed as a collection of vertices and edges which provides connection between input vertices and output vertices by nonintersecting (vertex-disjoint) paths. Various types of communication networks, such as concentrators, superconcentrators, generalizers, and rearrangeable and nonblocking networks, have been extensively studied[1-8] and can be used to build efficient switching networks or to serve as useful tools for complexity theory for algorithms.[8]

An (n, m)-concentrator is a graph with n input vertices and m output vertices, $n \geq m$, having the property that, for any set of m or fewer inputs, a set of vertex-disjoint paths exists that join the given inputs in a one-to-one fashion to different outputs. If this graph is directed or acyclic, we call it a directed or acyclic (n, m)-concentrator, respectively. Pinsker[4] shows the existence of a directed acyclic (n, m)-concentrator with $29n$ edges. We show that there exist (n, m)-concentrators with $15n$ edges, there exist directed (n, m)-concentrators with about $25n$ edges, and there exist directed acyclic (n, m)-concentrators with $27n$ edges.

An n-superconcentrator is a graph with n inputs and n outputs having the property that, for any set of inputs and any equinumerous set of outputs, a set of vertex-disjoint paths exists that join the given inputs in a one-to-one fashion to the given outputs. Valiant[9] shows the existence of directed acyclic n-superconcentrators with $238n$ edges.

Pippenger[5] improved this bound by showing the existence of directed acyclic n-superconcentrator with about $39n$ edges. We show there exist n-superconcentrators with about $18.5n$ edges, there exist directed superconcentrators with about $36n$ edges, and there exist directed acyclic superconcentrators with about $38.5n$ edges.

An n-generalizer is a graph with n inputs and n outputs having the property that, for any given correspondence between inputs and nonnegative integers that sum to n, a set of vertex-disjoint trees exists that join each input to the corresponding number of distinct outputs. Pippenger[7] proves that directed acyclic n-generalizers exist with about $120n$ edges. We show that n-generalizers exist with about $61.5n$ edges.

An n-nonblocking graph is a graph with n inputs and n outputs having the property that, for any given sequence of one-to-one correspondences between inputs and outputs, we can establish vertex-disjoint paths to join inputs to the corresponding outputs sequentially without disturbing existing paths. Bassalygo and Pinsker[1] prove the existence of directed and acyclic n-nonblocking graphs with $67.26n \log_2 n$ edges. Pippenger improved the bound to $56.79n$ edges. In this paper we show that directed acyclic n-blocking graphs exist with about $55n$ edges.

II. PRELIMINARIES

We first prove some auxiliary lemmas that mainly follow the lines of Pinsker[4] and Pippenger.[5]

Lemma 1: For integers n, a, b, x, $a \geq b \geq 2$ and a real number $\alpha < 1$, a bipartite graph exists with an inputs and bn outputs in which every input has degree bx and every output has degree ax and so that, for every set of $k \leq \alpha an$ inputs, a k-matching exists from the given inputs to some set of k outputs provided

$$x > \frac{H(\alpha) + (b/a)H(a\alpha/b)}{bH(\alpha) - \alpha a H(b/a)}$$

and

$$\alpha < \frac{b(b-2)}{ab - a - b} \quad \text{and} \quad xb > 4,$$

where

$$H(z) = -z \log_2 z - (1 - z)\log_2(1 - z)$$

is the well-known entropy function and n is sufficiently large.

Proof: For a permutation p on $\{0, 1, \cdots, abxn - 1\}$, we consider a labeled bipartite graph B_p with an inputs and bn outputs in which every input y is adjacent to the outputs $\{z{:}z \equiv p(y')(\text{mod } bn)$ for some

$y' \equiv y(\text{mod } an)\}$. The total number of such bipartite graphs is $(abxn)!$

Suppose B_p has the property that there are k inputs, $k \leq \alpha an$, such that there is no k-matching between these k inputs and some k outputs. From Hall's Theorem,[8,10] we know that there exist some k' inputs, $k' \leq k$, such that the total number of outputs adjacent to at least one of the k' inputs is less than k'. Thus the total number of B_p satisfying the above property does not exceed

$$A = \sum_{k \leq \alpha an} \binom{an}{k}\binom{bn}{k} \frac{(axk)!}{(axk - bxk)!} \cdot (abxn - bxk)!$$

Suppose $A < (abxn)!$ Then a bipartite graph B_p exists in which, for any k inputs, a k-matching exists between these given inputs and some k outputs. It suffices to show that $A < (abxn)!$ We let

$$f(k) = \binom{an}{k}\binom{bn}{k} \frac{(axk)!}{(axk - bxk)!} (abxn - bxk)!$$

For $k = \beta an$, we define

$$g(\beta) = \frac{f(k+1)}{f(k)}.$$

It is easily verified that

$$g(\beta) = (1 + o\,(1)) \cdot \frac{t^a}{(t-1)^{a-b}} \cdot \frac{(t^{-1} - \beta)\beta^{b-2}}{(1-\beta)^{b-1}},$$

where $t = a/b$ and $o(1)$ is arbitrarily small when n is sufficiently large. Let

$$p(\beta) = \frac{(t^{-1} - \beta)\beta^{b-2}}{(1-\beta)^{b-1}}.$$

By straightforward calculation, we have the first derivative $p'(\beta) > 0$ since

$$\beta < \alpha < \frac{b(b-2)}{ab - a - b}.$$

Let β_0, $0 < \beta_0 < \alpha$, be the real-number solution of

$$\frac{t^a}{(t-1)^{a-b}} \cdot p(\beta) = 1.$$

It follows immediately that, for n sufficiently large, we have

$$f(k+1) > f(k) \quad \text{if} \quad \beta = \frac{k}{an} > \beta_0 \quad \text{and}$$

$$f(k+1) < f(k) \quad \text{if} \quad \beta = \frac{k}{an} < \beta_0.$$

We consider the following two possibilities.

Case 1: $f(1) > f(\lfloor \alpha a n \rfloor)$.

From (1) we have

$$A \le \alpha a n f(1)$$

$$\le \alpha a^2 b n^3 \cdot \frac{(ax)!}{(ax - bx)!} (abxn - bx)!$$

$$< \frac{\alpha a^2 b n^3}{(bn - 1)^{bx}} \cdot (abxn)!$$

We have

$$A < (abxn)!$$

since $bx \ge 4$ and n is large.

Case 2: $f(\lfloor \alpha a n \rfloor) > f(1)$.

We use the following inequality for binomial coefficients (see Ref. 11).

$$((8np(1-p))^{-1/2})2^{nH(p)} \le \binom{n}{np} \le ((2\pi np(1-p))^{-1/2})2^{nH(p)}.$$

$$A \le \alpha a n f(\lfloor \alpha a n \rfloor)$$

$$\le \frac{\alpha a n \binom{an}{\alpha a n}\binom{bn}{\alpha a n}\binom{\alpha a^2 x n}{\alpha a b x n}}{\binom{abxn}{\alpha a b x n}} (abxn)!$$

$$< \frac{1}{\pi\sqrt{(1-(a\alpha)/b)(1-(b/a))}}\, 2^{anH(\alpha)} + bnH\left(\frac{a\alpha}{b}\right)$$

$$+ a^2 \alpha n x H\left(\frac{b}{a}\right) - abxnH(\alpha)(abxn)!$$

$$< (abxn)! \quad \text{for} \quad n \quad \text{sufficiently large.}$$

This completes the proof of Lemma 1.

The bipartite graphs in Lemma 1 will be denoted by $B(n, a, b, \alpha)$. We also let $\bar{B}(n, a, b, \alpha)$ denote the same bipartite graph except that the set of inputs and outputs are interchanged.

Lemma 2: *For any integers n and t and real numbers* α, β, $0 < \alpha < \beta < 1$, *a bipartite graph exists with n inputs and tn outputs in which* *every input has degree tx and every output has degree x and every set of k inputs, $k = \alpha\theta n < \alpha n$, are adjacent to at least $\beta\theta tn$ different outputs provided*

$$x > \frac{H(\alpha) + tH(\beta)}{t(H(\alpha) - \beta H(\alpha/\beta))}$$

and

$$xt > \frac{2 + \beta/\alpha + 3\alpha\beta}{1 - \beta}.$$

Proof: For a permutation p on $\{0, 1, \cdots, xtn - 1\}$, we consider a labeled bipartite graph B_p with n inputs and tn outputs in which every input y is adjacent to outputs $\{z{:}z \equiv p(y')(\mathrm{mod}\ tn)$ for some $y' \equiv y(\mathrm{mod}\ n)\}$. The total number of such bipartite graphs is $(xtn)!$

The number of B_p having the property that some k, $k = \alpha\theta n < \alpha n$, inputs are connected to less than $\beta\theta tn$ different outputs is bounded above by

$$A' = \sum_{k=\alpha\theta n<\alpha n} \binom{n}{\alpha\theta n}\binom{tn}{\beta\theta tn} \frac{(x\beta\theta tn)!(xtn - \alpha\theta xtn)!}{(x\beta\theta tn - x\alpha\theta tn)!}.$$

By an argument similar to that in the proof of Lemma 1, we can prove that

$$A' < (xtn)!$$

since

$$xt > \frac{2 + \beta/\alpha + 3\alpha\beta}{1 - \beta}$$

and

$$H(\alpha) + tH(\beta) + x\beta H\left(\frac{\alpha}{\beta}\right) - xtH(\alpha) < 0.$$

Therefore a bipartite graph B_p exists having the property that every set of k inputs, $k = \alpha\theta n < \alpha n$, are adjacent to at least $\beta\theta tn$ different outputs.

The bipartite graph mentioned in Lemma 2 will be denoted by $B'(n, t, \alpha, \beta)$.

III. SUPERCONCENTRATORS

A one-sided n-superconcentrator is a graph with n terminal vertices such that, for any two sets of equinumerous terminal vertices, we can find vertex-disjoint paths connecting vertices of one set to the vertices of the other.

Theorem 1: *There exist one-sided n-superconcentrators* $S_0(n)$ *with* $17.5n + O(\log n)$ *edges.*

Proof: Figure 1 illustrates the recursive construction for $S_0(n)$, where B is a subgraph of $B(\lceil n/7 \rceil, 7, 5, \frac{1}{2})$ in Lemma 1.

For any two sets of k terminal vertices, say, X and Y, we want to find vertex disjoint paths connecting vertices in X to vertices in Y. Let $X' = X - Y$, $Y' = Y - X$. It suffices to find vertex-disjoint paths connecting vertices in X' to vertices in Y'. We note that the number k' of vertices in X' does not exceed $n/2$. From Lemma 1, we know that a matching exists between X' and some set X'' of k' output vertices of B. Similarly, a matching exists between Y' and some set Y'' of k' output vertices of B. Vertices in X'' and Y'' can be connected by vertex-disjoint paths in $S_0(5\lceil n/7 \rceil)$. Therefore, $S_0(n)$ is indeed a superconcentrator.

The number of edges in B is $5n$. The number of edges in $S_0(n)$ is bounded above by

$$5n + 17.5\left(5\left\lceil\frac{n}{7}\right\rceil\right) + 80\left(\log 5\left\lceil\frac{n}{7}\right\rceil\right) \le 17.5n + 80(\log n).$$

Theorem 1 is proved.

Theorem 2: *There exists n-superconcentrator* $S(n)$ *with* $18.5n + O(\log n)$ *edges.*

Proof: The construction for $S(n)$ is shown in Fig. 2.

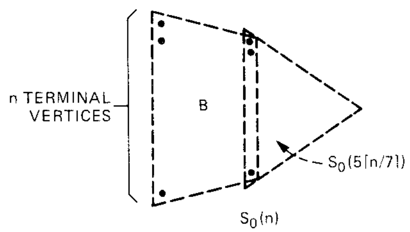

Fig. 1—A recursive construction for $S_0(n)$.

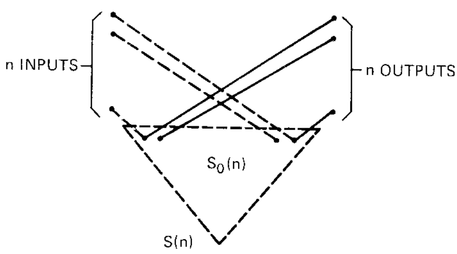

Fig. 2—A construction for $S(n)$.

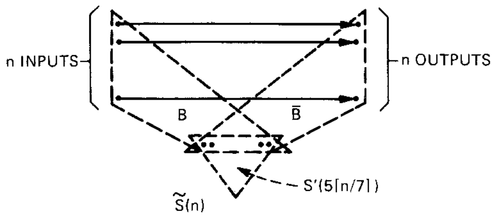

Fig. 3—A construction for $\tilde{S}(n)$.

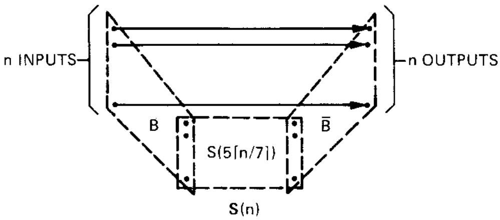

Fig. 4—A construction for $\mathbf{S}(n)$.

It is easily seen that $S(n)$ is a superconcentrator and has $18.5n + O(\log n)$ edges.

Theorem 3: *There exists directed superconcentrators* $\tilde{S}(n)$ *with* $36n + O(log\ n)$ *edges.*

Proof: The construction of $\tilde{S}(n)$ is shown in Fig. 3, where $S'(n)$ is a directed graph obtained by replacing each edge in $S_0(n)$ by two directed edges of different directions.

It is easily seen that $\tilde{S}(n)$ has $36n + O(\log n)$ edges.

Theorem 4: *There exist directed acyclic superconcentrators* $\mathbf{S}(n)$ *with* $38.5n + O(log\ n)$ *edges.*

Proof: $\mathbf{S}(n)$ is constructed similar to that in Ref. 4 except that the parameters are different (see Fig. 4).

It can be easily seen that $\mathbf{S}(n)$ has $38.5n + O(\log n)$ edges.

IV. CONCENTRATORS

An (n, m)-concentrator can be constructed as follows:

Case 1: $m < \dfrac{n}{3}$.

We construct $C(n, m)$ as shown in Fig. 5. For $1 \le i \le m$, input i is connected to output i by an edge. The connection between the inputs i, $i > m$, and the outputs can be viewed as a "composition" of two bipartite graphs, i.e., the outputs of the first bipartite graph are inputs of the second bipartite graph. The first bipartite graph is a subgraph B of $B(\lceil n/6 \rceil, 4, 3, 1/3)$ with $n - m$ inputs and $n/2$ outputs. From

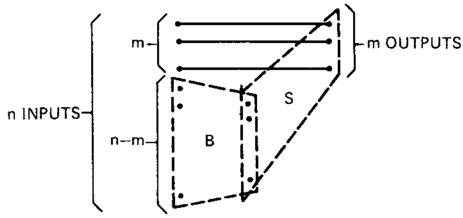

Fig. 5—A construction of $C(n, m)$ for $m < (n/3)$.

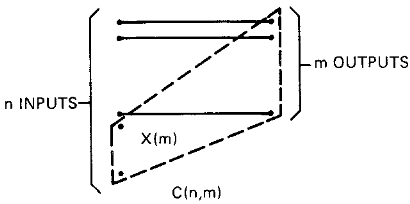

Fig. 6—A construction of $C(n, m)$ for $(n/3) < m \le (2/3)n$.

Lemma 1, we know that the first bipartite graph contains at most $4n$ edges. The second bipartite graph is a subgraph S of $S_0(\lceil n/2 \rceil)$ with $n/2$ inputs and m outputs.

To show that $C(n, m)$ in Fig. 5 is a concentrator, we let W denote a set of k inputs.

Suppose input i is in W and $i \le m$. We join input i to output i. Let W' be the set of inputs i, $i > m$, in W. From Lemma 1, we know that a matching exists between W' and some outputs in B, which will then be joined to specified distinct outputs since S is a subgraph of $S_0(\lceil n/2 \rceil)$. Therefore, $C(n, m)$ in Fig. 5 is a concentrator.

Case 2: $n/3 < m \le 2/3\ n$.

If we take $X(m)$ in Fig. 6 to be a subgraph of $2/3\ n$-superconcentrator with n-m inputs and m outputs, it is easily checked that $C(n, m)$ is a concentrator.

Case 3: $m > 2/3\ n$.

We choose B in Fig. 7 to be a subgraph of $\bar{B}(\lceil n/4 \rceil, 3, 2, 1/3)$ if $3/4\ n \ge m > 2/3\ n$, to be a subgraph of $\bar{B}(\lceil n/8 \rceil, 7, 4, 1/4)$ if $7/8\ n \ge m > 3/4\ n$, and to be a subgraph of $\bar{B}(n, 2, 1, 1/8)$ if $n \ge m > 7/8\ n$. In any of the three cases, the number of edges in B does not exceed $4n + 7$.

To see that $C(n, m)$ is a concentrator, we consider the case $7/8\ n > m \ge 3/8\ n$. The other two cases can be verified similarly. Now let A denote a set of $k \le m$ inputs. We want to show that a set of vertex-disjoint paths exists connecting vertices in A to some k output vertices. For any number x, $0 \le x < n - m$, we consider the set of inputs $I_x = \{y : y \equiv x (\text{mod } n - m)\}$ and the set of outputs $O_x = \{z : z \equiv x (\text{mod } n -$

$m)\}$. We note that $|I_x| \geq |O_x| \geq |I_x| - 1$. Suppose $|I_x \cap A| < |O_x|$. We can join vertices in $I_x \cap A$ to O_x by a matching. Thus we only have to consider the case that $|I_x \cap A| = |I_x| = |O_x| + 1$. The first $|O_x|$ vertices in $|I_x \cap A|$ can be connected to vertices in 0_x by a matching. Thus at least 3/4 m vertices in A can be connected to output vertices by vertex-disjoint edges. The remaining $\lfloor m/4 \rfloor$ vertices in A will then be connected through $S_0(\lceil n/2 \rceil)$. To see this, we note that B is a subgraph of $\bar{B}(\lceil n/8 \rceil, 7, 4, 1/4)$. For any set of $m/4$ output vertices of B, a matching exists between the given output vertices and some input vertices of B. Thus these vertices will be connected to the vertices in A through $S_0(\lceil n/2 \rceil)$.

Theorem 5: *There exist* (n, m)*-concentrators with* $14.75n + O(\log n)$ *edges.*

Proof: By the construction mentioned above, we note that in Case 2 we have

$$|C(n, m)| \leq m + |S(\lceil 2/3\, n \rceil)| \leq 13n + O(\log n).$$

In Case 1 and 3, we have

$$|C(n, m)| \leq n + 5n + |S_0(\lceil n/2 \rceil)| + O(\log n) \leq 14.75n + O(\log n).$$

Theorem 6: *There exist directed* (n, m)*-concentrators* $\tilde{C}(n, m)$ *with* $24.67n + O(\log n)$ *edges.*

Proof: By taking $X(n)$ to be $S'(m)$, we have

$$\tilde{C}(n, m) \leq 24.67n + O(\log n).$$

Theorem 7: *There exist directed acyclic* (n, m)*-concentrators* $\mathrm{C}(n, m)$ *with* $27n + O(\log n)$ *edges.*

Proof: By taking $X(m)$ to be the directed acyclic superconcentrator, we have $|\mathrm{C}(n, m)| \leq 27n + O(\log n)$.

Theorem 8: *There exists undirected, directed, acyclic directed* $(n, n/2)$*-concentrators with* $9.75n + O(\log n)$, $18.5n + O(\log n)$, $19.75n + O(\log n)$ *edges, respectively.*

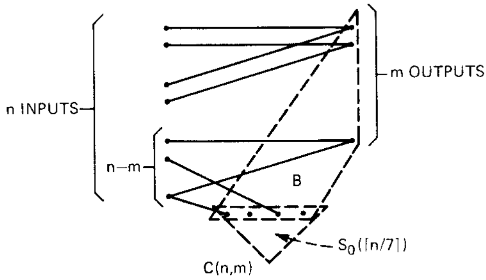

Fig. 7—A construction for $C(n, m)$ for $m > (2/3)n$.

Proof: We can construct undirected, directed, acyclic directed $(n, n/2)$-concentrator by taking S'' in Fig. 8 to be $S(\lceil n/2 \rceil)$, $\tilde{S}(\lceil n/2 \rceil)$ and $\mathbf{S}(\lceil n/2 \rceil)$, respectively. We can then obtain the desired bounds.

V. GENERALIZERS

An n generalizer can be constructed as follows (see Ref. 6 and Fig. 9):

$$|G(n)| \leq 3/2\, n + |G(n/2)| + |S(n)| + |S(\lceil n/2 \rceil)|.$$

Using Theorems 1 to 4, we have the following:

Theorem 9: *There exist n-generalizers with* $61.5n + O(\log n)$ *edges.*

Theorem 10: *There exist directed n-generalizers with* $111n + O(\log n)$ *edges.*

Theorem 11: *There exist directed acyclic n-generalizers with*

$$118.5n + O(\log n) \textit{ edges.}$$

VI. NONBLOCKING GRAPHS

A k-access graph $G(n, m, k)$ is a graph with n inputs, m outputs having the property that, for any given set S of vertex-disjoint paths connecting inputs to outputs, an input vertex which is not in S can be

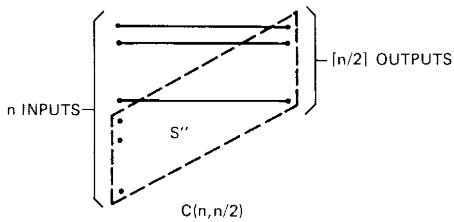

Fig. 8—A construction for $C(n, n/2)$.

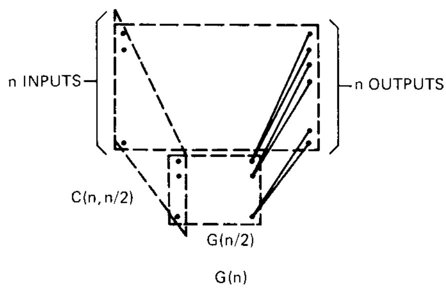

Fig. 9. A construction for $G(n)$.

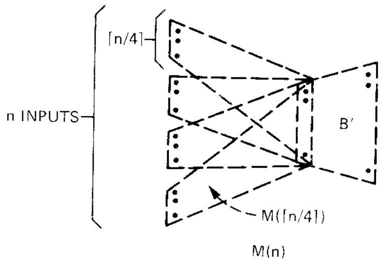

Fig. 10. A construction for $M(n)$.

connected to k different outputs by paths not containing any vertex in S. If k is greater than or equal to $m/2$, a k-access graph is also called a major-access graph. A nonblocking graph with n inputs and n outputs can be built by combining a major-access graph $G(n, m, k)$ and its mirror image.

We now construct a major access graph by using the $B'(n, t, \alpha, \beta)$ in Lemma 2 by a method similar to that in Ref. 6.

Let $M(n)$ denote a major access graph with n inputs, at most $f(n) = 11n + 17 \log_2 n$ outputs, constructed as shown in Fig. 10.

$M(n)$ will be a $2/3\, f(n)$-access graph if we take B' to be $B'(f(\lceil n/4 \rceil), 4, 13/33, 25/33)$. We note that B' has $20f(\lceil n/4 \rceil)$ edges. The total number of edges in $M(n)$ is

$$|M(n)| \leq |4M(\lceil n/4 \rceil)| + 220\lceil n/4 \rceil + 340 \log_2 n/4$$
$$\leq 27.4n \log_2 n + O((\log n)^2).$$

We will prove by induction on n that $M(n)$ is a $2/3\ f(n)$-access graph. For any given set S of vertex-disjoint paths connecting inputs to outputs in $M(n)$, let x denote an input vertex not in S. Suppose x is an input of M' which is a copy of $M(\lceil n/4 \rceil)$. We can connect x to $2/3\ f(\lceil n/4 \rceil)$ output vertices of $M(\lceil n/4 \rceil)$ by paths not containing any vertex in a path in S, which join a vertex of M' to some output of M'. Thus x is connected to about 13/33 of the total number of inputs of B' which are not in S. Thus x is connected to 25/33 of the number of outputs of B' of which at most n vertices can be in S. Therefore x is connected to 2/3 of the number of outputs of B' by paths not containing vertices in S. Therefore $M(n)$ is a $(2/3)\ f(n)$-access graph and we note that $M(n)$ is, in fact, a one-sided nonblocking graph. We have the following:

Theorem 12: *There exist one-sided nonblocking graphs with* $27.5n \log_2 n + O((\log n)^2)$ *edges.*

Theorem 13: *There exist directed acyclic nonblocking graphs* $N(n)$ *with* $55n \log_2 n + O((\log n)^2)$ *edges.*

Proof: The proof follows immediately from

$$|N(n)| \le |2M(n)|.$$

We note that, if we choose the size of $M(n)$ more carefully, say, $M(n)$ having $10.9n + O(\log n)$ outputs, we can, in fact, show that $|N(n)| \le 54.5n \log_2 n$.

VII. REMARKS

In this paper we prove the existence of various graphs by nonconstructive combinatorial probabilistic arguments. Margulis[3] gives an explicit construction for a sequence of bipartite graphs in which one bipartite graph satisfies the property required in Lemma 1 for given n, a, b, α, or in Lemma 2 for given n, α, β, t. However, he could not determine which one in the sequence is the bipartite graphs we need.

We remark that the bounds in Theorems 1 to 4 are the best possible under the conditions proven in Lemma 1 (by choosing a = 7, $b = 5$). We also note that all the bounds would be improved if we can improve the bound on x in Lemma 1 or 2.

In this paper, we deal with various graphs which represent corresponding switching networks. In the graph representation of a switching network, a point corresponds to a "line" and an edge corresponds to a "crosspoint." (The reader is referred to Ref. 2 for detail.) Therefore, the number of edges in the graph corresponds to the number of crosspoints which is a major part of the cost for a switching network. For example, the bounds we proved in Theorem 12 provide an estimate for the number of crosspoints of a nonblocking network. We summarize our results in Table I.

We remark that Ofer Gabber and Zvi Galil have recently found explicit constructions of linear size concentrators and superconcentrators. They proved the existence of an n-superconcentrator with $273n$ edges constructively by a complicated analytical argument. In fact, the constant 273 can be lowered to 262.

Table I

Graph \ No. of Edges \ Type	Undirected	Directed	Acyclic directed
One-sided superconcentrator	$17.5n + O(\log n)$	$35n + O(\log n)$	
Superconcentrator	$18.5n + O(\log n)$	$36n + O(\log n)$	$38.5n + O(\log n)$
Concentrator	$14.75n + O(\log n)$	$24.67n + O(\log n)$	$27n + O(\log n)$
$(n, n/2)$-concentrator	$9.75n + O(\log n)$	$18.5n + O(\log n)$	$19.5n + O(\log n)$
Generalizer	$61.5n + O(\log n)$	$111n + O(\log n)$	$118.5n + O(\log n)$
One-sided nonblocking graph	$27.25n \log_2 n$	$54.5n \log_2 n$	
Nonblocking graph	$54.5\ n \log_2 n$	$54.5n \log_2 n$	$54.5n \log_2 n$

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