THE CONNECTION PATTERNS OF TWO COMPLETE BINARY TREES*

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Abstract. We consider the class of channel graphs which can be viewed as compositions of two copies, right and left, of a complete binary tree with terminal nodes of the right tree connected to distinct terminal nodes of the left tree. We study the connection patterns of the two binary trees to minimize the blocking probability of the resulting channel graphs. We show that the connection patterns given by Ikeno are not optimal in general and in fact no optimal connection patterns exist for such graphs with more than 9 stages. We present new connection patterns which uniquely possess certain optimal properties.

1. Introduction. We consider the class of graphs which consist of two copies, right and left, of a complete binary tree with terminal nodes of the right tree connected to distinct terminal nodes of the left tree. If the complete binary trees have n levels and 2^{n-1} terminal nodes, then there are $(2^{n-1})!$ possible ways to connect these two trees, though some of them might be isomorphic. In Fig. 1 there are two nonisomorphic graphs, each of which is formed by combining two binary trees having three levels. In general such a graph can be viewed as a $2n$-stage network having the roots of the two complete binary trees as the source and sink of the network. We call such a graph a *binary channel graph* or a *binary graph*. We note that a binary graph is a special case of multistage *channel graphs,* also called *linear graphs,* which are often used in studying the blocking performance of switching networks [3]. A channel graph can be viewed as the union of all paths connecting a specified pair of input and output terminals in a switching network. In an *m-stage channel graph,* all vertices are partitioned into a sequence of m subsets, called *stages;* edges, called *links,* exist only between vertices in adjacent stages. The links between stage i and stage $i + 1$ are referred to as the ith stage links, and we assume that each ith stage link has probability p_i of being occupied. The vector $(p_1, \cdots, p_{m-1})$ is called the *link occupancies* of the channel graph. The *blocking probability* of a channel graph is defined to be the probability that every path from the source to the sink contains at least one occupied edge. Two channel graphs of the same number of stages can be compared in the following way. We say one channel graph is *superior* to the other if the blocking probability of the former never exceeds that of the latter for any given link occupancies. A $2n$-stage binary graph is said to be *optimal* if it is superior to any other $2n$-stage binary graph.

The problem of determining the connection pattern of the two binary trees to minimize the blocking probability of the resulting binary graph is not only interesting on its own right but also useful in designing effective switching networks (see [3]). Ikeno [1] investigated this problem and suggested the following simple and elegant connection: Assign binary numbers to terminal nodes of the right and left trees in an orderly fashion and connect two terminal nodes such that the digits of the corresponding binary numbers are inversions of each other. For example, the connections in the graph in Fig. 1(b) are (0, 0) to (0, 0), (0, 1) to (1, 0), (1, 0) to (0, 1) and (1, 1) to (1, 1).

It can be easily verified that the Ikeno graph is optimal for $n = 3$. However, the optimality for Ikeno graphs for $n > 3$ has not been previously determined in the past,

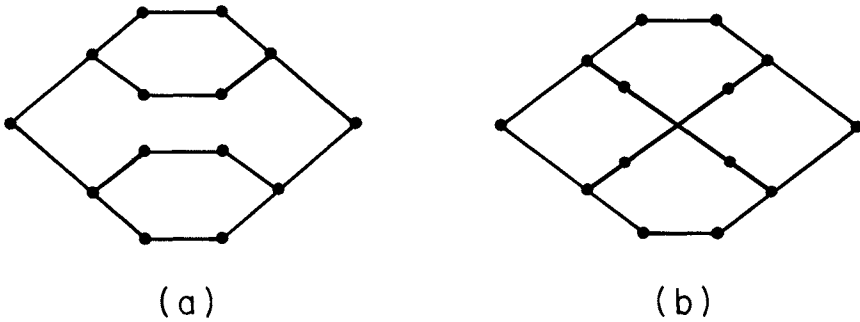---

(a)   (b)

FIG.1

though there has been a general willingness to believe it (see Neiman [3], for example). In this paper, we show that the optimal binary graph is not the Ikeno graph for $n = 4$, and for $n > 4$ no optimal binary graphs exist. We will give a connection pattern for $2n$-stage binary graphs which uniquely possess certain optimal properties.

2. Labelings and connection functions for binary graphs. We first give a labeling for the set of vertices of a binary graph. This labeling is useful in specifying the structure of a binary graph, in particular, in determining whether two binary graphs are isomorphic. Let T be a binary tree in $(n + 1)$ levels. We label the set of vertices of T except the root as follows:

(i) The set of vertices in the level $k + 1, k \geqq 1$, are labeled by the set of k-tuples with entries either 0 or 1, i.e., $\{l(u): u$ is a vertex in level $k + 1$ in $T\} = \{0, 1\}^k$.

(ii) Let u be a vertex in level $k + 1$ adjacent to a vertex v in level k. Then, the label of u, $l(u)$, is either $(l(v), 0)$ or $(l(v), 1)$ for $k \geqq 1$.

For a given $2(n + 1)$-stage binary graph, we label both the left tree and the right tree satisfying properties (i) and (ii). Then this binary graph can be characterized by a function $f: \{0, 1\}^n \to \{0, 1\}^n$ such that

$$f(a_1, \cdots, a_n) = (b_1, \cdots, b_n)$$

implies that the vertex with label $(a_1, \cdots, a_n)$ of the left tree is connected to the vertex with label $(b_1, \cdots, b_n)$ of the right tree. f is called *the connection function* of the binary graph. For example, the connection function of the graph in Fig. 1(a) is $f_1(a, b) = (a, b)$. The connection function of the graph in Fig. 1(b) is $f_2(a, b) = (b, a)$. Also, the $2(n + 1)$-stage Ikeno graph has connection function $f(a_1, \cdots, a_n) = (a_n, \cdots, a_1)$.

It is easily verified that f is a bijection (one-to-one and onto). For any given bijection $f: \{0, 1\}^n \to \{0, 1\}^n$, we can construct a binary graph having f as its connection function. There are $(2^n)!$ such bijections. However, two binary graphs constructed by using two different connection functions might be isomorphic. Thus, we need some methods for determining whether two binary graphs with distinct connection functions are isomorphic or not. We first introduce some terminology.

Let C be a vector of $2^n - 1$ elements; i.e., $C = \{c_{iS}: S \subseteq \{1, \cdots, i - 1\}, 1 \leqq i \leqq n\}$, where $c_{iS} \in \{0, 1\}$.

Let $\mathbf{C}$ denote the following function from $\{0, 1\}^n$ to $\{0, 1\}^n$, such that the ith coordinate of $\mathbf{C}(x_1, \cdots, x_n)$ is $x_i + \sum_{S \subseteq \{1, \cdots, i-1\}} c_{iS} \pi(S)$, where $\pi(S)$ denotes the product of all x in S, and $\pi(S) = 1$ if S is the empty set. We note that the numbers in the proceeding expression are in $\{0, 1\}$ reduced modulo 2.

LEMMA 1. *Two binary graphs, G_1 and G_2, having connection functions f_1 and f_2*

respectively, are isomorphic if and only if there exist two $(2^n - 1)$*-vectors* C, D *such that*

$$Cf_1 = f_2 D.$$

Proof. Suppose $Cf_1 = f_2 D$ for some vectors C and D. We define a map α from the vertex set of G_1 to the vertex set of G_2 as follows:

Let u be a vertex of the left tree in G_1 with label $(a_1, \cdots, a_i)$. Then we define $\alpha(u)$ to be the vertex of the left tree in G_2 with label $D(a_1, \cdots, a_i)$.

Let v be a vertex of the right tree in G_1 with label $(b_1, \cdots, b_i)$. Then we define $\alpha(v)$ to be the vertex of the right tree in G_2 with label $C(b_1, \cdots, b_i)$.

Moreover, the roots of the left tree and the right tree of G_1 are mapped to the roots of the left tree and right tree of G_2 respectively.

It is easily seen that α is one to one and onto. By straightforward verification based on the fact that $Cf_1 = f_2 D$, it can be shown that for $u, v \in V(G_1)$, u is adjacent to v if and only if $\alpha(u)$ is adjacent to $\alpha(v)$ in G_2. Thus α is an isomorphism, and the two graphs G_1 and G_2 are isomorphic.

Now, we assume G_1 and G_2 are isomorphic. f_1 and f_2 can be viewed as connection functions determined by two labelings l_1 and l_2 of the same binary graph G. Let us first define another labeling l_3 of G such that $l_3(u) = l_1(u)$ for any u in the left tree of G and $l_3(v) = l_2(v)$ for any v in the right tree of G. Let f_3 denote the connection function of G determined by the labeling l_3.

CLAIM. $f_3 = Cf_1$ *for some* $(2^n - 1)$*-vector* C.

Let us first consider all the labelings of G with vertices in the left tree labeled as in l_1. There are exactly 2^{2^n-1} ways to label the right tree. However, for any vector $C = \{c_{iS} \colon S \subseteq \{1, \cdots, i - 1\}, 1 \leq i \leq n, c_{iS} \in \{0, 1\}\}$, we can define a labeling of the vertices in the right tree of G as follows:

$$l_C(v) = C(b_1, \cdots, b_i),$$

where $l_1(v) = (b_1, \cdots, b_i)$.

We note that all labelings l_C, $C \in \{0, 1\}^{2^n-1}$ are distinct. Thus $\{l_C \colon C \in \{0, 1\}^{2^n-1}\}$ are exactly the set of all labelings of vertices in the right tree of G. Therefore, there exists a vector C such that $l_3(v) = l_C(v)$ for any vertex v in the right tree.

Thus, for a terminal vertex u in the left tree which is adjacent to a terminal vertex v in the right tree, we have

$$f_1(l_1(u)) = l_1(v),$$

$$f_3(l_3(u)) = l_3(v) = l_C(v) = Cl_1(v) = Cf_1(l_1(u)).$$

Since $l_1(u) = l_3(u)$ for any u in the left tree, we have

$$f_3 = Cf_1.$$

In a similar way it can be shown that $f_3 = f_2 D$ for some vector D. Thus we obtain $Cf_1 = f_2 D$ and the lemma is proved.

Lemma 1 is, in fact, equivalent to the following.

COROLLARY. *Two* $2(n + 1)$*-stage binary graphs,* G_1 *and* G_2*, having connection functions* f_1 *and* f_2*, respectively, are isomorphic if and only if there exist two* $(2^n - 1)$*-vectors* C *and* E *such that*

$$f_2 = Cf_1 E.$$

Proof. It suffices to show that the inverse function of the linear transformation D

is of the form $\mathbf{E}$ for some vector E. This, however, can be done by straightforward calculation to find values of e_{ij}, $1 \leqq j \leqq i \leqq n$ such that $\mathbf{DE} = \mathbf{I}$ where $\mathbf{I}$ is the identity transformation.

Let f_1, f_2 be two bijections from $\{0, 1\}^n$ to $\{0, 1\}^n$. We say f_1 is *equivalent* to f_2 if there exists two $(2^n - 1)$-vectors, say C and D such that

$$f_1 = \mathbf{C}f_2\mathbf{D}.$$

We note that two connections functions derived from different labelings of a binary graph are equivalent.

3. The optimal 8-stage binary graph. It can be easily verified that the Ikeno graph is an optimal $2n$-stage binary graph for $n = 1$, 2 and 3. However, as an immediate result of the following theorem the Ikeno graph is not optimal for $n = 4$.

THEOREM 1. *The optimal 8-stage binary graph G_8 is isomorphic to the binary graph determined by the connection function $f(a_1, a_2, a_3) = (a_3, a_2 + a_1a_3, a_1)$ as shown in* Fig. 2.

Before we prove Theorem 1, we state a result of Takagi [6]. Let T_k denote the class of 4-stage channel graphs where there are k each of the second-stage and the third-stage vertices, and each second-stage (third-stage) vertex is connected to two third-stage (second-stage) vertices.

THEOREM (*Takagi*). *A graph in T_k is optimal if and only if the $2k$ middle-stage links form a cycle.*

Proof of Theorem 1. Let L_i denote the set of ith stage links. It suffices to show that in each of the following four cases, the blocking probability of G_8 achieves minimum (since the four cases are mutually disjoint and exhaustive).

(i) Both links in L_1 and both links in L_7 are idle. In this case, an 8-stage binary graph is not blocked if there exists a path from any second-stage vertex to any seventh-stage vertex. Therefore the graph can be viewed as a 6-stage graph by eliminating the first and last stage and combining vertices in stage 2 and stage 7. By *shrinking* a graph from stage i to stage j we mean replacing every path between the two stages by a link. When all these paths are edge disjoint, then by simply defining the probability of the new link being busy as the probability of the path being busy, the blocking probability of the shrunken graph is the same as the blocking probability of the original graph. By shrinking the reduced G_8 from stage 2 to stage 5, we obtain a 4-stage graph in T_4. Fig. 3 shows such a T_4 obtained from G_8:
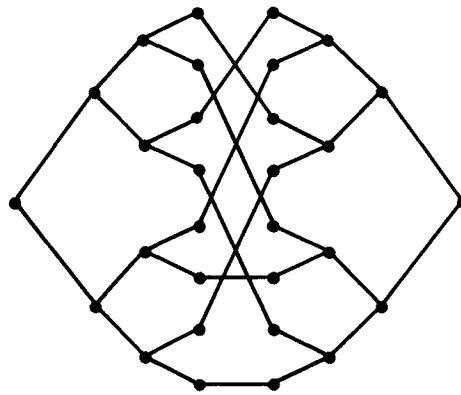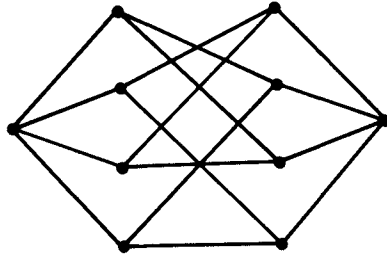

FIG. 2

FIG. 3

Note that the eight middle-stage links in this graph form a cycle. Theorem 1 follows immediately from the Takagi Theorem.

(ii) Both links in L_1 and one link in L_7 are idle. G_8 is reduced to the 6-stage graph in Fig. 4. It is straightforward to verify that this is the best reduction.

(iii) One link in L_1 and both links in L_7 are idle. Same as (ii).

(iv) One link in L_1 and one link in L_7 are idle. We have to consider the set of four reduced graphs corresponding to the four possible combinations of idle links. There are three possible sets for 8-stage binary graphs. A Type A set consists of two graphs of four paths and two empty graphs. A Type B set consist of two graphs of three paths and two graphs of one path. A Type C set consists of four graphs of two paths. The reduced G_8 is a Type C set and it is straightforward to verify that the sum of blocking probabilities of the four graphs in a Type C set is minimal.

From (i), (ii), (iii) and (iv), we conclude that G_8 is an optimal 8-stage graph. It is straightforward to verify that G_8 is indeed the unique optimal 8-stage graph up to isomorphism since there are only a few distinct connections to check. We omit the details here.

The 8-stage Ikeno graph I_8 is illustrated in Fig. 5. It can be easily seen that G_8 is superior to the 8-stage Ikeno graph I_8 since the blocking probability of G_8 is less than or equal to the blocking probability of I_8 in case (i); and the blocking probability of G_8 is equal to the blocking probability of I_8 in case (ii), (iii) and (iv) for any given link occupancies.

4. Some preliminary results on binary graphs. In a channel graph G, let u be a vertex in stage i and v a vertex in stage j where $j > i$. The *channel subgraph* determined by u and v in G is defined to be the union of all paths of length $j - i$ connecting u and v in G. A *maximal channel subgraph* of G is a channel subgraph of G having the number of stages one less than the number of stages in G. A maximal channel subgraph is usually determined by a vertex in the second stage and the sink (the vertex in the last stage) or the source (the vertex in the first stage) and a vertex in the stage next to the last.
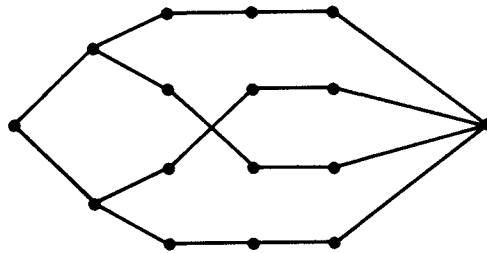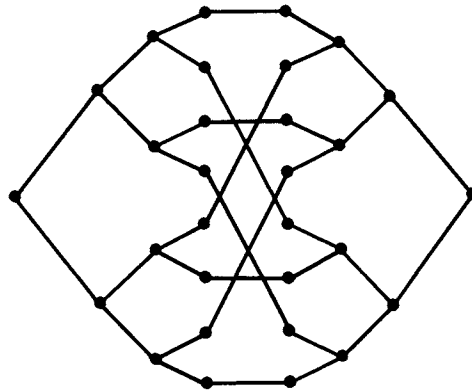

FIG. 4

FIG. 5

We say a maximal channel subgraph of a $2n$-stage binary graph is optimal if it is isomorphic to an optimal $2(n - 1)$-stage binary graph after shrinking from stage $n - 1$ to stage $n + 1$ or from stage n to stage $n + 2$.

LEMMA 2. *All the maximal channel subgraphs of an optimal $2n$-stage binary graph are optimal.*

Proof. Let G' be a maximal channel subgraph determined by a vertex in stage 2 and the sink in the optimal $2n$-stage binary graph G. Let H be a $2n$-stage binary graph such that all its maximal channel subgraphs determined by a vertex in stage 2 and the sink are, after shrinking from stage n to stage $n + 2$, isomorphic to a given arbitrary $2(n - 1)$-stage binary graph H'. To be precise, let f denote the connection function of a $2(n - 1)$-stage binary graph. We can then construct H with the connection function f^* to be $f^*(a_1, \cdots, a_{n-1}) = (b_1, \cdots, b_{n-1})$, where $f(a_2, \cdots, a_{n-1}) = (b_1, \cdots, b_{n-2})$ and $a_1 = b_n$. Now we consider the link occupancies $p_1, \cdots, p_{n-1}$ where p_1 is extremely small compared with p_i, $i \neq 1$. Then the blocking probabilities of G and H can be approximated by the product of p_1 and the blocking probability of G' and H', respectively. Since G is an optimal binary graph, G' must be optimal.

COROLLARY. *If no optimal $2n$-stage binary graph exists, then no optimal $2m$-stage binary graph exists for $m \geqq n$.*

In a later section we will show that optimal 10-stage binary graphs do not exist. Thus optimal $2n$-stage binary graphs, $n \geqq 5$, do not exist. It is then of interest to investigate binary graphs with certain optimal properties.

We note that the optimal $2n$-stage binary graphs, for $n \leqq 4$, have the *symmetric* property, i.e., the connection function f satisfies the following:

$$f(a_1, \cdots, a_{n-1}) = (b_1, \cdots, b_{n-1}), \quad \text{if and only if}$$

$$f(b_1, \cdots, b_{n-1}) = (a_1, \cdots, a_{n-1}).$$

In other words, the connection function f is *idempotent*, $f = f^{-1}$. Symmetry might be a desirable property for practical considerations since it facilitates the construction and the control of the network.

We define G_8 to be an HS-optimal 8-stage binary graph (it is in fact optimal). For $n \geqq 5$, a $2n$-stage binary graph is said to be *hereditary* if all its maximal channel subgraphs, after shrinking, are $2(n - 1)$-stage HS-optimal binary graphs. We call a $2n$-stage binary graph HS-optimal if it is both hereditary and symmetric. The main result of this paper is to prove that for $n \geqq 5$, there exists a unique HS-optimal $2n$-stage

binary graph. In this section, we give a set of lemmas concerning HS-optimal $2n$-stage binary graphs.

From the definition of HS-optimal binary graphs, the following observations are immediate.

LEMMA 3. *Let f be the connection function of an HS-optimal $2(n + 1)$-stage binary graph. We define the function $\bar{f}_{a_1}$: $\{0, 1\}^{n-1} \to \{0, 1\}^{n-1}$ by the following:*

$$\bar{f}_{a_1}(a_2, \cdots, a_n) = (b_1, \cdots, b_{n-1}),$$

where $f(a_1, a_2, \cdots, a_n) = (b_1, \cdots, b_n)$. Then $\bar{f}_{a_1}$ is the connection function of the maximal channel subgraph determined by the sink and the vertex of the left tree with labeling (a_1). Similarly $(\bar{f}^{-1})_{b_1}$ is the connection function of the maximal channel subgraph determined by the source and the vertex of the right tree with labeling (b_1).

LEMMA 4. *A $2(n + 1)$-stage binary graph having connection function f is hereditary if and only if $\bar{f}_a$ and $(\bar{f}^{-1})_b$, for $a, b \in \{0, 1\}$, are equivalent to connection functions of an HS-optimal $2n$-stage binary graph.*

Proof. This follows from the definition of hereditary.

LEMMA 5. *Let f be the connection function of an HS-optimal $2(n + 1)$-stage binary graph G. Then f is equivalent to the following function g:*

$$g(a_1, \cdots, a_n) = (\bar{f}_{a_1}(a_2, \cdots, a_n), a_1).$$

Proof. It suffices to show that we can choose a proper labeling of G with connection function g. Because G is HS-optimal, for any labeling l of G the two vertices with labeling $(b_1, \cdots, b_{n-1}, 0)$ and $(b_1, \cdots, b_{n-1}, 1)$ are connected to two vertices in stage $n + 1$ with distinct first components, i.e.,

$$f(a_1, \cdots, a_n) = (b_1, \cdots, b_{n-1}, 0)$$

and

$$f(a_1', \cdots, a_n') = (b_1, \cdots, b_{n-1}, 1)$$

imply $a_1 \neq a_1'$.

Thus we consider a labeling l' of G such that a vertex v with $l(v) = (b_1, \cdots, b_{n-1}, b_n)$ will be labeled as $l'(v) = (b_1, \cdots, b_{n-1}, b_n + 1)$, if $a_1 \neq b_n$, and $l'(v) = (b_1, \cdots, b_n)$ otherwise, where $f(a_1, \cdots, a_n) = (b_1, \cdots, b_n)$. We note that the connection function g under the labeling l' can be written as follows:

$$g(a_1, \cdots, a_n) = (\bar{f}_{a_1}(a_2, \cdots, a_n), a_1).$$

Since f and g are connection functions of G in two labelings, we know f and g are equivalent from Lemma 2.

Thus Lemma 5 is proved.

LEMMA 6. *Let x be a number depending on a where $x, a \in \{0, 1\}$. Then x can be written as follows:*

$$x = aw_1 + w_2 \qquad (\mathrm{mod}\ 2),$$

where w_1, w_2 are some constants in $\{0, 1\}$.

Proof. It is easy to find w_1, w_2 by solving two equations for $a = 0$ or 1. Namely, $w_2 = x(0)$, $w_1 = x(1) - x(0)$.

For any vector $C = \{c_{iS}\}$ we let $\mathbf{C}_a$ denote the linear transformation determined by $\{c_{iS}a\}$.

LEMMA 7. *Let C be a $(2^n - 1)$-vector $\{c_{iS}: S \subseteq \{1, \cdots, i - 1\}, i = 1, \cdots, n\}$ where c_{iS}'s depend on a. Then the linear transformation $\mathbf{C}$ can be written as follows:*

$$C = DE = FG$$

where E and F are $(2^n - 1)$-vectors (independent of a) and $D = D_a$, $G = G_a$.

Proof. The values of elements of $\mathbf{E}$, and $\mathbf{D}$ can be obtained by using Lemma 6 and solving the $n(n + 1)$ linear equations derived from $\mathbf{C} = \mathbf{DE}$. The values of elements of $\mathbf{F}$ and $\mathbf{G}$ can be obtained similarly.

LEMMA 8. *Let G be a $2(n + 1)$-stage hereditary binary graph. There exists a labeling of G such that the connection function can be written as follows:*

(*) $$f(a_1, \cdots, a_n) = (\mathbf{M}_{a_1} g \mathbf{N}_{a_1}(a_2, \cdots, a_n), a_1),$$

where g is some connection function of an HS-optimal 2n-stage binary graph, and $n_{i\phi} = 0$, for $i = 1, \cdots, n - 1$.

Proof. Let h be the connection function of a $2(n + 1)$-stage hereditary binary graph G. From Lemma 5 we may assume

$$h(a_1, \cdots, a_n) = (b_1, \cdots, b_n) = (\bar{h}_{a_1}(a_2, \cdots, a_n), a_1).$$

Since G is hereditary, we know that, by Lemma 4, $\bar{h}_{a_1}(a_2, \cdots, a_n) = (b_1, \cdots, b_{n-1})$ is a connection function of the $2n$-stage HS-optimal binary graph. By the induction assumption, we have $\bar{h}_a = \mathbf{R}g\mathbf{S}$ where the elements of R or S depend on a. Therefore, by Lemma 7, we have

$$\mathbf{R} = \mathbf{UM}_a,$$

and

$$\mathbf{S} = \mathbf{N}'_a \mathbf{V},$$

where elements of U and V are independent of a. We define U' and V' such that $\mathbf{U}'(x_1, \cdots, x_n) = (\mathbf{U}(x_1, \cdots, x_{n-1}), x_n)$ $\mathbf{V}'(x_1, \cdots, x_n) = (x_1, \mathbf{V}(x_2, \cdots, x_n))$, and we note that we can find a $(2^n - 1)$-vector W such that $\mathbf{W}(x_1, \cdots, x_n) = (x_1, \mathbf{W}'_{x_1}(x_2, \cdots, x_n))$ and $\mathbf{N}'_{a_1} = \mathbf{N}_{a_1} \mathbf{W}'_{a_1}$, where $n_{i\phi} = 0$ for $i = 1, \cdots, n$.

It is easy to verify that

$$\mathbf{U}'f\mathbf{WV}' = h,$$

where $f(a_1, \cdots, a_n) = (\mathbf{M}_{a_1} g \mathbf{N}_{a_1}(a_2, \cdots, a_n), a_1)$. Therefore, Lemma 8 is proved.

From now on, for any HS-optimal binary graph, we will only consider its connection functions in form as described in Lemma 8. We note that several examples of connection functions we give in this paper are in this form.

LEMMA 9. *A $2(n + 1)$-stage binary graph G is HS-optimal if and only if the connection function of G is equivalent to f satisfying*

(1)
$$f(a_1, \cdots, a_n) = (\mathbf{M}_{a_1} g \mathbf{N}_{a_1}(a_2, \cdots, a_n), a_1)$$
$$= (a_n, (\mathbf{N}_{a_n})^{-1} g^{-1} (\mathbf{M}_{a_n})^{-1}(a_1, \cdots, a_{n-1})),$$

where g is a connection function of the optimal 2n-stage binary graph and M and N are two $(2^n - 1)$-vectors.

Proof. Since G is symmetric and hereditary, we have (1) from Lemma 8. Now, suppose G has a connection function equivalent to f. It follows immediately G is symmetric. By Lemma 3 and 4, G is also hereditary. Thus G is HS-optimal.

Now we consider a set of connection functions $P = \{p_i: i = 1, 2, \cdots\}$ where $p_i: [0, 1]^i \rightarrow [0, 1]^i$. We set

$$p_2(a_1, a_2) = (a_2, a_1),$$

$$p_3(a_1, a_2, a_3) = (a_3, a_2 + a_1 a_3, a_1) \qquad (\text{mod } 2).$$

The p_i for $i \geqq 4$, will be defined in §§ 5, 6.

We note that p_2, p_3 are connection functions for optimal 6- and 8-stage binary graphs. The set P denotes the set of the connection functions for HS-optimal binary graphs.

5. 10-stage binary graphs. For 10-stage binary graphs, we will show the following.

THEOREM 2. *The HS-optimal 10-stage binary graph has connection function (up to isomorphism) as follows (see Fig. 6):*

$$p_4(a_1, a_2, a_3, a_4) = (a_4, a_3 + a_2 a_4, a_2 + a_1 a_3 + a_1 a_2 a_4, a_1) \qquad (\text{mod } 2).$$

We remind the reader that all calculations are in Z_2. We will, from now on, omit the notation (mod 2).

Proof. It is straightforward to verify that p_4 is idempotent.

The function p_4 can be written as

$$p_4(a_1, a_2, a_3, a_4) = (\mathbf{L}_{a_1} p_3(a_2, a_3, a_4), a_1)$$

$$= (a_4, p_3 \mathbf{L}_{a_4}(a_1, a_2, a_3)),$$

where

$$\mathbf{L}_a(x_1, x_2, x_3) = (x_1, x_2, x_3 + ax_2)$$

$$= (\mathbf{L}_a)^{-1}(x_1, x_2, x_3).$$

It follows from Lemma 9 that the binary graph having p_4 as the connection function is HS-optimal.

Now, suppose G is an HS-optimal 10-stage binary graph having connection function f. From Lemma 9 f satisfies the following

(2) $$f(a_1, a_2, a_3, a_4) = (\mathbf{M}_{a_1} p_3 \mathbf{N}_{a_1}(a_2, a_3, a_4), a_1)$$

(3) $$= (a_4, (\mathbf{N}_{a_4})^{-1} p_3 (\mathbf{M}_{a_4})^{-1}(a_1, a_2, a_3)).$$
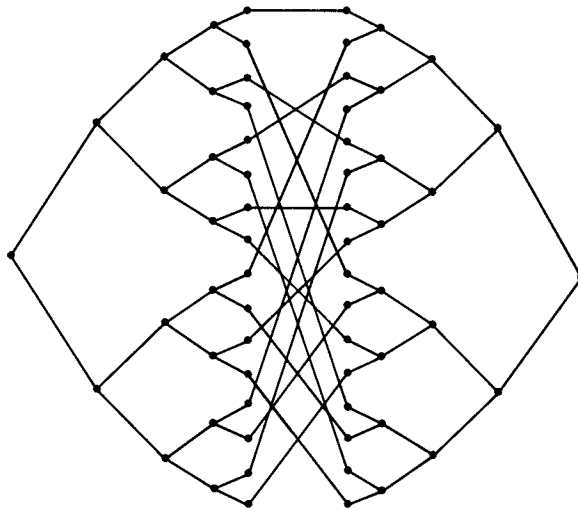

FIG. 6

To simplify the notation we denote the sets $\{1\}, \{2\}, \{1, 2\}$ by 1, 2, 12, respectively. Equating the values in the first coordinates of (2) and (3), we obtain

$$a_4 = a_4 + a_1 a_2 n_{3,1} + a_1 a_3 n_{3,2} + a_1 a_2 a_3 n_{3,12} + a_1 m_{1\phi},$$

$$\text{for any} \quad a_i \in \{0, 1\}, \ 1 \leqq i \leqq 4.$$

Thus, we have $m_{1\phi} = n_{3,1} = n_{3,2} = n_{3,12} = 0$.

Suppose we replace a_4 by 0. The value of the second coordinates in (2) and (3) is

$$a_3 = a_3 + m_{2\phi} a_1 + n_{2,1} a_1 a_2.$$

Therefore $m_{2\phi} = n_{2,1} = 0$ and $\mathbf{N}$ is then the identity function. Let us write

$$\mathbf{M}(x_1, x_2, x_3) = (x_1, x_2 + m_{2,1} x_1, x_3 + m_{3\phi} + m_{3,1} x_1 + m_{3,2} x_2 + m_{3,12} x_1 x_2).$$

Then we have

$$\mathbf{M}^{-1}(x_1, x_2, x_3) = (x_1, x_2 + m_{2,1} x_1, x_3 + m_{3\phi}$$

$$+ (m_{3,1} + m_{2,1} m_{3,2} + m_{2,1} m_{3,12}) x_1 + m_{3,2} x_2 + m_{3,12} x_1 x_2).$$

We consider the value of the second coordinates in (2) and (3). We have

$$a_3 + a_2 a_4 + m_{2,1} a_1 a_4 = a_3 + m_{3\phi} a_4 + (m_{3,1} + m_{2,1} m_{3,2} + m_{2,1} m_{3,12}) a_4 a_1$$

$$+ m_{3,2} a_2 a_4 + m_{3,12} a_1 a_2 a_4.$$

Therefore we have $m_{3,1} = m_{3\phi} = m_{3,12} = 0$, $m_{3,2} = 1$.

Now we consider the third coordinate in (2) and (3). We have $a_2 + a_1 a_3 + a_1 a_2 a_4 = a_2 + m_{2,1} a_1 a_4 + a_1 a_3 + a_1 a_2 a_4$. Therefore we have $m_{2,1} = 0$, $\mathbf{M} = \mathbf{L}$ and $f = p_4$. Theorem 2 is proved.

It should be noted that the 10-stage HS-optimal binary graph is not optimal since in the case that $p_1 = p_2 = p_8 = p_9 = 0$, then the graph can be reduced and shrunken to a 4-stage graph in T_8 (similar to what we did to G_8 in the proof of Theorem 1). However, the 16 middle-stage links form two cycles instead of one. Since there does exist a 10-stage binary graph which, after similar reductions and shrinking, yields a graph in T_8 with the middle-stage links forming a cycle, then by Takagi's Theorem the HS-optimal 10-stage binary graph is not optimal. Furthermore, it can be shown (by straightforward arguments) that there exists only one (up to isomorphism) hereditary graph which is not symmetric. Again by using Takagi's Theorem it can be seen that this graph is not optimal. However, by Lemma 2, an optimal 10-stage binary graph must be hereditary. Therefore, no optimal 10-stage binary graph exists.

6. $2n$-stage binary graphs for $n \geqq 5$. The main result in this paper is the following.

THEOREM 3. *The HS-optimal $(2n + 2)$-stage binary graph has the connection function (up to isomorphism) as follows:*

$$p_n(a_1, \cdots, a_n) = (b_1, \cdots, b_n),$$

where

$$b_1 = a_n,$$

$$b_i = a_{n+i-1} + a_{n-i} b_{i-1} \quad (\text{mod } 2), \text{ for } i = 2, \cdots, n - 1,$$

$$b_n = a_1.$$

Proof. First we define

$$\mathbf{L}(x_1, \cdots, x_i) = (x_1, \cdots, x_{i-1}, x_i + x_{i-1}).$$

It can be verified by induction that

$$p_n(a_1, \cdots, a_n) = (\mathbf{L}_{a_1} p_{n-1}(a_2, \cdots, a_n), a_1)$$

$$= (a_n, p_{n-1} \mathbf{L}_{a_n}(a_1, \cdots, a_{n-1})).$$

The binary graph determined by p_n is thus symmetric and hereditary.

Now we consider a connection function g of HS-optimal $2(n + 1)$-stage binary graph G. From Lemmas 8 and 9, g is equivalent to an f which has the following form

(4) $$f(a_1, \cdots, a_n) = (\mathbf{M}_{a_1} p_{n-1} \mathbf{N}_{a_1}(a_2, \cdots, a_n), a_1)$$

(5) $$= (a_n, (\mathbf{N}_{a_n})^{-1} p_{n-1} (\mathbf{M}_{a_n})^{-1}(a_1, \cdots, a_{n-1})).$$

We want to show by induction on n that:

(*):   if f can be written in the form of (4) and (5), then $\mathbf{N}$ is the identity mapping and $\mathbf{M} = \mathbf{L}$.

(*) is true for $n = 4$ from Theorem 2. Suppose (*) is true for all n' with $4 \leq n' < n$. We will prove that (*) holds for n.

The value in the first coordinates in (4) and (5) is

$$a_n = a_n + \sum_{S \subseteq \{0, \cdots, n-1\}} c_{n,S} \pi(S), \quad \text{for all} \quad a_i \in \{0, 1\}, 1 \leq i \leq n.$$

Thus we have $c_{n,S} = 0$, for all S.

We consider, $\mathbf{M}'$, $\mathbf{N}'$ and $\mathbf{H}$ to be mappings from $\{0, 1\}^{n-2} \to \{0, 1\}^{n-2}$ such that

$$\mathbf{M}(0, x_2, \cdots, x_{n-1}) = (0, \mathbf{M}'(x_2, \cdots, x_{n-1})),$$

$$\mathbf{N}(x_1, \cdots, x_{n-1}) = (\mathbf{N}'(x_1, \cdots, x_{n-2}), x_{n-1}),$$

$$\mathbf{H}(x_1, \cdots, x_{n-2}) = (x_1 + m_{21}, x_2 + m_{31}, \cdots, x_{n-2} + m_{n-1,1}).$$

It is easy to see that

$$\mathbf{M}(x_1, \cdots, x_{n-1}) = (x_1, \mathbf{H}_{x_1} \mathbf{M}'(x_2, \cdots, x_{n-1})),$$

$$\mathbf{M}^{-1}(x_1, \cdots, x_{n-1}) = (x_1, \mathbf{M}'^{-1} \mathbf{H}_{x_1}(x_2, \cdots, x_{n-1})),$$

$$\mathbf{N}^{-1}(x_1, \cdots, x_{n-1}) = (\mathbf{N}'^{-1}(x_1, \cdots, x_{n-2}), x_{n-1}).$$

By induction $p_{n-1}(x_1, \cdots, x_{n-1}) = (x_{n-1}, p_{n-2} \mathbf{L}_{x_{n-1}}(x_1, \cdots, x_{n-2}))$. Therefore (4) is equal to the following:

$$((\mathbf{M}_{a_1}(a_n, p_{n-2} \mathbf{L}_{a_n} \mathbf{N}'_{a_1}(a_2, \cdots, a_{n-1}))), a_1)$$

$$= (a_n, \mathbf{H}_{a_1 a_n} \mathbf{M}'_{a_1} p_{n-2} \mathbf{L}_{a_n} \mathbf{N}'_{a_1}(a_2, \cdots, a_{n-1}), a_1).$$

On the other hand, (5) is equal to the following:

$$(a_n, \mathbf{N}_{a_n}^{-1}(\mathbf{L}_{a_1} p_{n-2} \mathbf{M}'^{-1}_{a_n} \mathbf{H}_{a_1 a_n}(a_2, \cdots, a_{n-1}), a_1))$$

$$= (a_n, \mathbf{N}_{a_n}^{-1} \mathbf{L}_{a_1} p_{n-2} \mathbf{M}'^{-1}_{a_n} \mathbf{H}_{a_1 a_n}(a_2, \cdots, a_{n-1}), a_1).$$

Therefore we have

(6) $$\mathbf{H}_{a_1 a_n} \mathbf{M}'_{a_1} p_{n-2} \mathbf{L}_{a_n} \mathbf{N}'_{a_1} = \mathbf{N}_{a_n}^{-1} \mathbf{L}_{a_1} p_{n-2} \mathbf{M}'^{-1}_{a_n} \mathbf{H}_{a_1 a_n}.$$

By setting $a_n = 0$ in (6), we have

$$\mathbf{M}'_a p_{n-2} \mathbf{N}'_{a_1} = \mathbf{L}_a p_{n-2}.$$

From the above equality and the definition of p_{n-2} and $\mathbf{L}$ we define f' as follows:

$$
\begin{aligned}
f'(x_1, \cdots, x_{n-1}) &= (\mathbf{M}'_{x_1} p_{n-2} \mathbf{N}'_{x_1}(x_2, \cdots, x_{n-1}), x_1) \\
&= (\mathbf{L}_{x_1} p_{n-2}(x_2, \cdots, x_{n-1}), x_1) \\
&= (x_{n-1}, p_{n-2} \mathbf{L}_{x_{n-1}}(x_1, \cdots, x_{n-2})) \\
&= (x_{n-1}, \mathbf{N}'^{-1}_{x_{n-1}} p_{n-1} \mathbf{M}'^{-1}_{x_{n-1}}(x_1, \cdots, x_{n-2})).
\end{aligned}
$$

By the induction assumptions, we have $\mathbf{M}'_x = \mathbf{L}_x$ and $\mathbf{N}'_x$ is the identity mapping. Therefore $\mathbf{N}$ is the identity mapping and (6) is equivalent to the following:

(7) $$\mathbf{H}_{a_1 a_n} \mathbf{L}_{a_1} p_{n-2} \mathbf{L}_{a_n} = \mathbf{L}_{a_1} p_{n-2} \mathbf{L}_{a_n} \mathbf{H}_{a_1 a_n}.$$

By setting $a_1 = a_n = a$ in (7), we have

$$\mathbf{H}_a \mathbf{L}_a p_{n-2} \mathbf{L}_a = \mathbf{L}_a p_{n-2} \mathbf{L}_a \mathbf{H}_a,$$

i.e.,

$$\mathbf{H}_a \mathbf{L}_a p_{n-2} \mathbf{L}_a \mathbf{H}_a \mathbf{L}_a = \mathbf{L}_a p_{n-2}.$$

Now we consider f'' as follows:

$$
\begin{aligned}
f''(x_1, \cdots, x_{n-1}) &= (\mathbf{H}_{x_1} \mathbf{L}_{x_1} p_{n-2} \mathbf{L}_{x_1} \mathbf{H}_{x_1} \mathbf{L}_{x_1}(x_2, \cdots, x_{n-1}), x_1) \\
&= (\mathbf{L}_{x_1} p_{n-2}(x_2, \cdots, x_{n-1}), x_1) \\
&= (x_{n-1}, p_{n-2} \mathbf{L}_{x_{n-1}}(x_1, \cdots, x_{n-2})) \\
&= (x_{n-1}, \mathbf{L}_{x_{n-1}} \mathbf{H}_{x_{n-1}} \mathbf{L}_{x_{n-1}} p_{n-2} \mathbf{L}_{x_{n-1}} \mathbf{H}_{x_{n-1}}(x_1, \cdots, x_{n-2})).
\end{aligned}
$$

As before, we have, by the induction assumptions, $\mathbf{H}_a \mathbf{L}_a = \mathbf{L}_a$ and $\mathbf{L}_a \mathbf{H}_a \mathbf{L}_a$ is the identity. This implies $\mathbf{H}_a$ is the identity and $\mathbf{M} = \mathbf{L}$. Therefore we have

$$
\begin{aligned}
f(a_1, \cdots, a_n) &= (\mathbf{L}_{a_1} p_{n-1}(a_2, \cdots, a_n), a_1) = (a_n, p_{n-1} \mathbf{L}_{a_n}(a_1, \cdots, a_{n-1})) \\
&= (b_1, \cdots, b_n).
\end{aligned}
$$

Theorem 3 is then proved.

7. Construction of switching networks with prescribed channel graphs. We will present several constructions of switching networks with prescribed channel graphs.
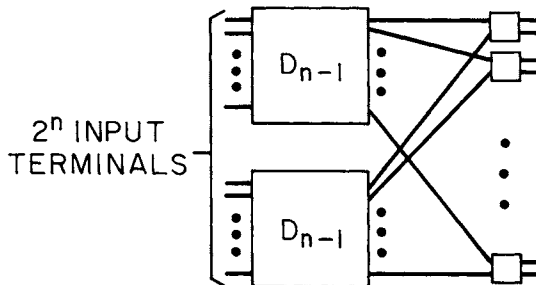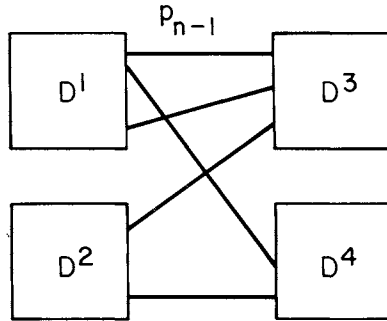

Fig. 7

FIG. 8

Let us first consider a $2n$-stage network N_{2n} consisting of 2×2 switches. Let D_1 denote a 2×2 switch and D_n be the network built recursively as shown in Fig. 7. The $2n$-stage network N_{2n} consists of 2 copies of D_n denoted by D^1, D^2 together with 2 copies denoted by D^3 and D^4, of the mirror image of D_n as shown in Fig. 8.

The linking pattern between the middle two stages is determined by the connection function of the channel graph p_{n-1}. Namely, the ith output switch of D_1 or D_2 is connected to the jth input switch of D^3 or D^4 if and only if $i - 1$ and $j - 1$ have the binary expressions $(a_1, \cdots, a_{n-1})$, $(b_1, \cdots, b_{n-1})$ respectively, and $p_{n-1}(a_1, \cdots, a_{n-1}) = (b_1, \cdots, b_{n-1})$.

It can be easily verified that the channel graph of N_{2n} is the HS-optimal binary graph G_{2n}.

Let us now consider networks with an odd number of stages. We note that a binary graph of $2n + 2$ stages can be viewed as a binary graph of $2n + 1$ stages by shrinking the two middle stages. We can then derive HS-optimal binary graphs G_{2n+1} from the HS-optimal graphs G_{2n+2}. For example G_5 is as shown in Fig. 9.

Now, we want to construct a $2n + 1$ stage network with channel graph G_{2n+1}. G_{2n+1} consists of a copy of D_{n+1} together with two copies, denoted by D^3, D^4, of the mirror image of D_n as shown in Fig. 10.

The linking pattern between stage $n + 1$ and stage $n + 2$ is determined by p_n. Namely, the ith output switch of D_{n+1} is connected to the jth input line of D^3 or D^4 if and only if $i - 1$ and $j - 1$ have the binary expressions $(a_1, \cdots, a_n)$, $(b_1, \cdots, b_n)$, respectively, and $p_n(a_1, \cdots, a_n) = (b_1, \cdots, b_n)$. It is easily verified that N_{2n+1} has channel graph G_{2n+1}.

8. Concluding remarks. We can generalize the ideas in this paper and consider the connection pattern of two t-ary graphs. A $(2n + 2)$-stage Ikeno graph is the t-ary graph determined by the connection function $f(x_1, \cdots, x_n) = (x_n, \cdots, x_1)$
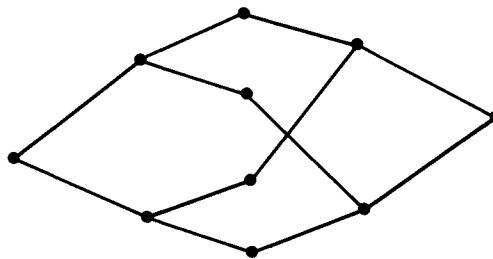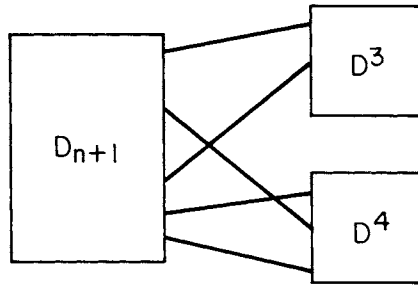

FIG. 9

FIG. 10

where the leaves of the left and right t-ary trees are labeled by elements in $\{0, \cdots, t - 1\}$. Again it can be shown that the Ikeno graph is not optimal for $t \geqq 3$, $n \geqq 3$ by considering an alternative connection function defined in a similar way as we defined p_n in previous sections. The connection patterns of t-ary graphs are probably more complicated than the binary case.

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