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ON THE CUTWIDTH AND THE TOPOLOGICAL BANDWIDTH OF A TREE*

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Abstract. We investigate the relations between the topological bandwidth $b^*(G)$ and the cutwidth $f(G)$ for a graph G. We show that for any tree T we have $b^* \leqslant f(T) \leqslant b^*(T)+\log_2 b^*(T)+2$. These bounds are "almost" best possible, since we will prove that for each n, there exists a tree T_n such that $b^*(T_n) = n$ and $f(T_n) \geqslant n+\log_2 n-1$, and the star S_{2n} with 2n edges satisfies $b^*(S_{2n}) = f(S_{2n}) = n$.

1. Introduction. Suppose G is a graph with vertex set $V(G)$ and edge set $E(G)$. A numbering π of G is a one-to-one mapping from $V(G)$ to the set of positive integers. Such a numbering can be viewed as describing a placement of the vertices of G on a line, so it is not surprising that graph numbering problems are frequently relevant to circuit layout and design. The following objective functions will be of interest in this paper.

(i) The bandwidth $b_\pi(G)$ of a numbering π is defined to be

$$b_\pi(G) = \max\{|\pi(u)-\pi(v)|: \{u,v\} \in E(G)\}$$

and the bandwidth $b(G)$ of G is the minimum of $b_\pi(G)$ over all numberings π of G.

(ii) The topological bandwidth $b^*(G)$ of a graph G is defined to be

$$b^*(G) = \min\{b(G'): G' \text{ is a refinement of } G\}$$

(A graph G' is said to be a refinement of G if G' is obtained from G by a finite number of edge subdivisions.)

(iii) Define

$$f_\pi(G) = \max_i |\{\{u,v\} \in E(G): \pi(u) \leqslant i < \pi(v)\}| .$$

Then the cutwidth [12] $f(G)$ of a graph G is defined to be

$$f(G) = \min_\pi f_\pi(G).$$

We will show that for any tree T the following holds:

$$b^*(T) \leqslant f(T) \leqslant b^*(T)+\log_2 b^*(T)+2 .$$

These bounds are "almost" best possible, since we will prove that for each n, there exists a tree T_n such that $b^*(T_n) = n$ and $f(T_n) \geqslant n+\log_2 n-1$, and the star S_{2n} with $2n$ edges satisfies $b^*(S_{2n}) = f(S_{2n}) = n$.

We remark that the upper bound does not hold for general graphs since for the complete graph K_n on n vertices we have $b^*(K_n) = n-1$ and $f(K_n) = \lceil (n^2-1)/4 \rceil$, though it can be shown that $b^*(G) \leqslant f(G)$ for general graphs G.

(A numbering of a graph is also called a linear arrangement of a graph [6]. The cutwidth of a graph is sometimes called the folding number of a graph [2].)

As to the algorithmic aspects, the bandwidth problem for graphs is known to be

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NP-complete [6], [9] as is the bandwidth problem for trees [5]. The cutwidth problem for graphs is also NP-complete [4], while the cutwidth problem for trees can be solved in $O(n \log n)$ time [13] (also see [3] for degree restricted cases). The topological bandwidth problem for graphs is recently proved to be NP-complete [8].

We remark that the minimum sum problem of finding $\min_\pi \sum_{\{u,v\}\in E(G)} |\pi(u)-\pi(v)|$ is NP-complete for graphs [8] while there are polynomial time algorithms for the minimum sum problem for trees [7].

2. Preliminaries. In this section we will discuss several properties of numberings [2] that will be useful later.

Let π denote a numbering of a tree T mapping $V(T)$ to $\{1,...,n\}$ where $n = |V(T)|$. We say π satisfies

(i) The leaf property, if the vertices numbered by 1 and n are leaves.

(ii) The monotone property, if the following is true: Let P denote the path, called the basic path of π, in T connecting the two vertices numbered by 1 and n. Suppose P has vertices $v_0,v_1,...,v_t$ with v_i adjacent to v_{i+1}. Then π is monotone if the numberings of the vertices of P are monotone, i.e.,

$$\pi(v_i) < \pi(v_{i+1}) \text{ for } v = 0,1,...,t-1 \text{ or}$$
$$\pi(v_i) > \pi(v_{i+1}) \text{ for } v = 0,1,...,t-1 .$$

(iii) The block property, if the following is true: Let F denote the forest formed by removing the edges of P from T (but let the vertices stay). Then any maximal tree in F is numbered by a set of consecutive integers.

(iv) The weak block property, if the following is true: Let $\bar{T}$ denote a maximal subtree in F. Suppose $x = \min\{\pi(u): u \in \bar{T}\}$ and $\underline{y} = \max\{\pi(u): u \in \bar{T}\}$. Then any vertex v with $x \leqslant \pi(v) \leqslant y$ is either in $\bar{T}$ or on P.

(v) The hereditary property, if the induced numbering for each subtree $\bar{T}$ of F is an optimal numbering with respect to the objective function of interest. (The induced numbering π' of π on T' is the one-to-one mapping from $V(T')$ to the set $\{1,2,...,|V(T')|\}$ such that for any $\{u,v\}$ in $E(T')$, $\pi'(u) < \pi'(v)$ if $\pi(u) < \pi(v)$. π' is denoted by π/T'.)

It is easy to check that for a given tree T there exists a bandwidth numbering π with $b_\pi(T) = \underline{b}(T)$ satisfying the leaf property. Also there exists a numbering $\bar{\pi}$ for a refinement $\bar{T}$ of T with $b_{\bar{\pi}}(\bar{T}) = b^*(T)$ satisfying the leaf property, the monotone property, and the weak block property. There always exists a cutwidth numbering λ with $f_\lambda(T) = f(T)$ satisfying the leaf property, the monotone property, the block property and the hereditary property.

Let π denote a numbering for a tree T. Then for any subtree T' in T, the basic path $P(\pi,T')$ of T' is the path joining the two vertices with the largest and smallest numbers in T'. Let $F(\pi,T,1)$ denote the forest obtained by removing the edges (not the vertices) of $P(\pi,T)$ from T. Let $F(\pi,T,i)$ denote the forest obtained by removing the edges of the basic paths of all maximal subtrees in $F(\pi,T,i-1)$. Then we have the following:

LEMMA 1. *Suppose λ is a cutwidth numbering for T. Then $f(T) = 1 + \max_{T'} f(T')$ for T' ranging over all maximal subtrees of $F(\lambda,T,1)$.*

Proof. This follows immediately from the monotone property, the block property and the hereditary property of λ.

LEMMA 2. *Suppose λ is a cutwidth numbering for T. Then $f(T) = i + \max_{T'} f(T')$ for T' ranging over all maximal subtrees of $F(\lambda,T,i)$.*

LEMMA 3. *If T' is a refinement of T, then we have*

$$f(T) = f(T') .$$

Proof. This follows from the fact that any numbering π of T can be extended to be a numbering $\bar{\pi}$ of T' with $f_\pi(T) = f_{\pi'}(T)$. On the other hand, for any numbering $\bar{\pi}$ of T' the induced numbering $\bar{\pi}/T$ of $\bar{\pi}$ on T satisfies $f_{\bar{\pi}/T}(T) \leqslant f_{\bar{\pi}}(T')$.

LEMMA 4. $f(T) \leqslant |V(T)|/2$.

Proof. This follows from the leaf property that any maximal subtree in $F(\lambda,F,1)$ has at most $|V(T)|-2$ vertices. Thus by Lemma 1 and by induction on $n = |V(T)|$ we have

$$f(T) = 1 + \max_{T'} f(T') \leqslant 1 + \frac{|V(T)|-2}{2} = \frac{|V(T)|}{2} .$$

LEMMA 5. *Suppose T' is a refinement of T. Then $b^*(T')$ can be different from $b^*(T)$.* (See Fig. 1.)

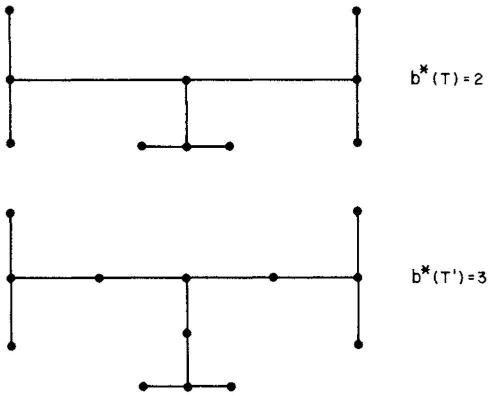

Fig. 1

Let us now define two functions, called the shifting function and the skipping function, from the set of integers Z to itself. The shifting function is $s_a(n) = a+n$ and the skipping function k_a is the order preserving function from Z to $Z - \{ia: i \in Z\}$.

LEMMA 6. *Suppose T is a tree which is the edge-disjoint union of a path P and a collection S of trees, say the ith vertex in P is in the ith tree in S. Then we have*

$$b^*(T) \leqslant 1 + \max_{T' \in S} b^*(T') .$$

Proof. Let $T_1,\dots,T_t$ denote the trees in S. Let T_i' be a refinement of T_i with a labeling π_i: $V(T_i') \to \{1,\dots,|V(T_i')|\}$ and $b_{\pi_i}(T_i') = b^*(T_i)$. We will combine the π_i to form a numbering π' for a refinement T' of T with $b_{\pi'}(T') = 1 + \max_i b^*(T_i) = 1+x$. Roughly speaking, the vertices of T_i' in T' are numbered in the same fashion as π_i except that the assigned values skip one out of every $x+1$ values. The numbering π' restricted to T_i' can be described as $s_{a_i}k_{x+1}\pi_i$ where

$$a_i = \lceil(\sum_{j<i}|V(T_i)|)(1+\frac{1}{x})\rceil + i|V(t)| .$$

Now we refine the basic path so that its vertices are numbered by a chain of numbers at most $x+1$ apart. Therefore we have

$$b^*(T) \leqslant b_\pi(T') = 1+x .$$

This completes the proof of Lemma 6.

LEMMA 7. *Suppose π is a numbering for a tree T and π satisfies the leaf property. Then we have*

$$b^*(T) \leqslant 1 + \max_{T'} b^*(T')$$

for T' ranging over all maximal subtrees in $F(\pi,T,1)$.

Proof. It follows from Lemma 6.

Let $F^*(\pi,T,1)$ denote the forest obtained by removing all vertices and edges in $P(\pi,T)$. Then we have the following.

LEMMA 8. *Suppose π is a bandwidth numbering of T. Then*

$$b(T) \geqslant 1 + \max_{T' \in F^*(\pi,T,1)} b(T') .$$

Proof. Suppose $b(T) = x$. For any vertex v in T with $\pi(v)+x \leqslant |V(T)|$ there is a vertex u in $P(\pi,T)$ such that $\pi(v) \leqslant \pi(u) \leqslant \pi(v)+x$. Thus for any T' in $F^*(\pi,T,1)$ the induced numbering of π on T' has bandwidth at most $x-1$.

3. The topological bandwidth is no larger than the cutwidth. It is easy to show that the topological bandwidth is no larger than the cutwidth numbering for a tree.

THEOREM 1. *$f(T) \geqslant b^*(T)$ for any tree T.*

Proof. We will prove this by induction on $|V(T)|$. Let λ denote the cutwidth numbering. Let T' denote a maximal subtree in $F(\lambda,T,1)$. We have

$$\begin{aligned} f(T) &\geqslant 1 + \max_{T'} f(T') && \text{(by Lemma 1)} \\ &\geqslant 1 + \max_{T'} b^*(T') && \text{(by induction and } |V(T')| < |V(T)|) , \\ &\geqslant b^*(T) && \text{(by Lemma 7)} . \end{aligned}$$

In fact, the topological bandwidth for a graph is no larger than its cutwidth. This has been observed by I. H. Sudborough and F. Makedon [11] among others. We will give the proof here.

THEOREM 2. *$f(G) \geqslant b^*(G)$ for any graph G.*

Proof. Let λ denote a cutwidth numbering of G. We will modify λ to obtain a numbering λ' of a refinement G' of G such that $b_{\pi'}(G') \leqslant f_\pi(G) = f(G) = x$. First we choose a subgraph G_1 of G as follows

Step 1: Set $C = \phi$.

Step 2: Choose an edge $\{u,v\}$ such that $\pi(u) \leqslant \pi(v)$ and u is the smallest vertex with $\pi(u) \geqslant \pi(w)$ for any w in a edge in C. Put $\{u,v\}$ into C and repeat Step 2. If no such edge exists, stop the process.

Clearly, the graph G_1 formed by edges in C has $f_\pi(G_1) = 1$. Also the graph $G-G_1$ obtained by removing edges in G_1 from G satisfies $f_\pi(G-G_1) = x-1$. (Otherwise, let i be the least number with $|\{\{u,v\} \in E(G-G_1): \pi(u) \leqslant i < \pi(v)\}| = x$. Then all edges $\{u,v\}$ in G with $\pi(u) \leqslant i < \pi(v)$ are not in G_1. From Step 2 we know that there is no edge $\{u,v\}$ in G with $\pi(u) = i < \pi(v)$. Thus there are x edges $\{u,v\}$ with $\pi(u) < i < \pi(v)$. This implies $|\{\{u,v\} \in \pi(G-G_1): \pi(u) \leqslant i-1 < \pi(v)\}| = x$, contradicting the minimality of i.)

We can then repeat the process and partition G into $G_1, G_2, \ldots, G_x$, such that $f_\pi(G_i) = 1$ for $1 \leqslant i \leqslant x$. Now we consider a refinement G' of G as follows. For any edge $\{u,v\}$ in G_i with $\pi(u) < \pi(v)$, we subdivide $\{u,v\}$ into a path of $\pi(v) - \pi(u) + 1$ vertices, $u = u_0, u_1, \ldots, u_t = v$ where $t = \pi(v) - \pi(u)$. We define $\pi'(u_j)$ to be $x(\pi(u)+j) + i-1$.

Clearly π' is a one-to-one function from $V(G')$ to Z. It is easily checked that $b_{\pi'}(G') = x$. Thus we have $f(G) = x = b_{\pi'}(G') \geqslant b^*(G)$.

4. The topological bandwidth for a tree is not equal to its cutwidth in general. For each integer n, we will construct a tree T_n satisfying $b^*(T_n) = n$ and $f(T_n) \geqslant n + \log_2 n - 1$. We will recursively build a rooted tree T_n^* (i.e., a tree with one special vertex) as follows: (i) T_1^* is a path with three vertices. The middle vertex is the root. (ii) For $n > 1$, T_n^* consists of a path P_n of 15 vertices and 15 copies of T_{n-1}^*. Each vertex in P_n is adjacent to the root of a copy of T_{n-1}^*. The root of T_n^* is the root of the T_{n-1}^* which is connected to the 8th vertex of P_n.

Let T_n denote the unrooted version of T_n^*.

CLAIM 1. $b^*(T_n) = n$.

Proof. We will prove this by induction on n. It is easily seen that $b^*(T_1) = 1$. Suppose a refinement $\bar{T}_i$ of T_i has bandwidth $\leqslant i$. We want to show that $b^*(T_{i+1}) = i+1$. Let π denote the numbering with (the refined) P_{i+1} as the basic path. Let T' denote a maximal subtree in $F^*(\pi, \bar{T}, 1)$. Then $T' \subseteq T_i$.

$$\begin{aligned} b^*(T_{i+1}) &\leqslant 1 + \max_{T'} b^*(T') \quad \text{(by Lemma 7)} \\ &\leqslant 1 + b^*(T_i) \leqslant 1+i \end{aligned}$$

On the other hand, for any topological-bandwidth numbering π of T_{i+1}, $F^*(\pi, T_{i+1}, 1)$ must contain T_i. Thus we have

$$\begin{aligned} b^*(T_{i+1}) &\geqslant 1 + \max_{T' \in F^*(\pi,T,1)} b^*(T') \quad \text{(by Lemma 8)} \\ &\geqslant 1 + b^*(T_i) \\ &\geqslant 1+i\,. \end{aligned}$$

Thus we have $b^*(T_{i+1}) = 1+i$.

CLAIM 2. $f(T_n) \geqslant n + \log_2 n - 1$.

Proof. This will be proved by induction on n. It is easy to see that $f(T_1) = 1$ and $f(T_2) = 3$. Suppose $f(T_j^*) \geqslant j + (1 + 1/j)\log_2 j - 1$ for $2 \leqslant j < i$. We want to prove $f(T_i^*) \geqslant i + (1 + 1/i)\log_2 i - 1$. Let π_i denote a cutwidth numbering of T_i. We say π_i is good if $P(\pi_i, T_i)$ contains at least 9 vertices of P_n. If π_i is good, then $F(\pi_i, T_i, 1)$ contains the tree which is the union of T_{i-1}^* and an edge incident to the root, denoted by $\tilde{T}_{i-1}$. Consider the restricted mapping π_{i-1} of π_i to $\tilde{T}_{i-1}$. For each j if π_{i-j} is good (i.e., $P(\pi_{i-j}, \tilde{T}_{i-j})$ contains 9 vertices of P_{n-j}), we consider $\tilde{T}_{i-j-1}$ (which is the union of T_{i-j-1}^* and $j+1$ additional edges incident to the root of T_{i-j-1}^*) and the restricted mapping π_{i-j-1} of π_{i-j} to T_{i-j-1}^* until π_{i-j_0} is not good. There are two possibilities.

CASE 1. $j_0 \leqslant i/2 + \log_2 i$ and $j_0 < i$. Since π_{i-j_0} is not good, $F(\pi_{i-j_0}, \tilde{T}_{i-j_0}, 1)$

contains a tree consisting of a path of length 3 joining to three copies of $T^*_{i-j_0-1}$. Thus $F(\pi_{i-j_0},\tilde{T}_{i-j_0},2)$ still contains a copy of $T^*_{i-j_0-1}$. We then have

$$f_{\pi_{i-j_0}}(\tilde{T}_{i-j_0}) \geqslant 2+f(T^*_{i-j_0-1})$$

and, by induction,

$$\begin{aligned} f(T_i^*) = f_{\pi_i}(T_i^*) &\geqslant j_0+f_{\pi_{i-j_0}}(\tilde{T}_{i-j_0}) \\ &\geqslant j_0+2+f(T^*_{i-j_0-1}) \\ &\geqslant j_0+2+i-j_0-1+(1+\frac{2}{i-j_0-1})\log_2(i-j_0-1)-1 \\ &\geqslant 1+i+(1+\frac{2}{i/2-\log_2 i-1})\log_2(\frac{i}{2}-\log i-1)-1 \\ &\geqslant i+(1+\frac{2}{i})\log_2 i-1. \end{aligned}$$

CASE 2. $j_0 > i/2 + \log_2 i$ or $j_0 = i$. Then $f(T_i^*) \geqslant j_0+f_{\pi_{i-j_0}}(\tilde{T}_{i-j_0})$. Note that $\tilde{T}_{i-j_0}$ contains a star S_{i+1} of $i+1$ edges. Thus

$$\begin{aligned} f(T_i^*) &\geqslant j_0 + f(S_{i+1}) \\ &\geqslant j_0 + \lceil\frac{i+1}{2}\rceil \\ &\geqslant i/2 + \log_2 i + \lceil\frac{i+1}{2}\rceil \\ &\geqslant i + \log_2 i + \frac{1}{2}. \end{aligned}$$

Therefore we have proved the following.

THEOREM 3. *For every positive integer n there exists a tree T satisfying*

$$\begin{aligned} &b^*(T) = n \quad \textit{and} \\ &f(T) \geqslant b^*(T)+\log_2 b^*(T)-1 . \end{aligned}$$

5. The difference between the topological bandwidth and the cutwidth for a tree is small. In this section, we will prove that the topological bandwidth for a tree can be bounded above by the sum of its cutwidth and a lower order term. The proof is somewhat complicated. We will give a sequence of observations from which the proof will follow. Suppose π is a bandwidth numbering. Let T' denote a maximal tree in $F(\pi,T,1)$. The numbering induced by π on T' has many special properties. Before we consider these helpful properties we will make some definitions.

Let π denote a numbering of T. We say π is an (x,y)-numbering of T if there is a multi-set $J(T)$ of y vertices (not necessarily distinct) of $V(T)$ such that for any edge $\{u,v\} \in E(T)$ with $\pi(u) < \pi(v)$, we have

$$|\pi(u)-\pi(v)| \leqslant x+|\{w\in J: \pi(u) < \pi(w) < \pi(v)\}| .$$

Furthermore, we say π is derived from a $(x+y,0)$-numbering $\bar{\pi}$ of $\bar{T}$ if π is the induced numbering of $\bar{\pi}$ on T for some $\bar{T}$ containing T. A tree having a (x,y)-numbering is a (x,y)-tree.

OBSERVATION 1. If the bandwidth of a tree T is x, then T is a $(x,0)$-tree.

OBSERVATION 2. Suppose π is a $(x,0)$-numbering of T and π satisfies the leaf property. Let T' denote a maximal tree in $F(\pi,T,1)$. Then T' is a $(x-1,1)$-tree while $J(T')$ is $V(T') \cap P(\pi,T)$.

Proof. For any value a with $1 \leqslant a < a+x \leqslant |V(T)|$ the set $\{u \in V(T)\colon a < \pi(u) \leqslant a+x\}$ contains at least one vertex in $P(\pi,T)$, as does the set $\{u \in V(T)\colon a \leqslant \pi(u) < a+x\}$. Thus the induced numbering π' of π on T' satisfies the property that for $\{u,v\} \in E(T')$ with $\pi(u) < \pi(v)$ we have

$$|\pi'(u)-\pi'(v)| \leqslant x-1+|\{u'\colon \pi(u) < \pi(u') < \pi(v')\} \cap u_0|$$

where $u_0 = V(T') \cap P(\pi,T)$, since $|\{u,v\} \cap P(\pi,T)| \leqslant 1$.

OBSERVATION 3. Suppose T has a $(x,0)$-numbering. Then there is a refinement $\bar{T}$ of T having $\underline{a}$ $(x,0)$-numbering $\bar{\pi}$ such that for each i and each maximal subtree T' in $F(\bar{\pi},\bar{T},i)$ the induced numbering π' on T' satisfies the leaf property, the monotone property, and the weak block property.

Proof. This follows from the fact that we can untangle the maximal trees.

From now on we will only consider $(x,0)$-numberings satisfying the properties in Observation 3.

OBSERVATION 4. Suppose T has a $(x,0)$-numbering. Then there is a refinement $\bar{T}$ of T having a $(x,0)$-numbering $\bar{\pi}$ such that for each i all the trees T' in $F(\bar{\pi},\bar{T},i)$ are $(x-i,i)$-trees.

Proof. For any value a with $\min_{v \in V(T')}\pi(v) \leqslant a < a+x \leqslant \max_{u \in V(T)}\pi(u)$, the set $\{u \in V(T)\colon a \leqslant \pi(u) < a+x\}$ contains at least one vertex in each basic path $P(\pi,T_j)$, $p \leqslant j \leqslant i$, $T_j \in F(\pi,T,j)$. Thus the induced numbering π' of π of T' satisfies the property that for $\{u,v\} \in E(T')$ with $\pi(u) < \pi(v)$, we have

$$|\pi'(u)-\pi'(v)| \leqslant x-i+|\{u'\colon \pi(u) < \pi(u') < \pi(v)\} \cap J(T')|$$

where $J(T')$ is the multi-set $\bar{\bigcup}_j(V(T') \cap P(\pi,T_j)$ $(\{a\} \mathbin{\bar{\bigcup}} \{a\}$ is defined to be $\{a,a\})$.

From now on we will only be interested in the (x,y)-numberings satisfying the leaf property, the monotone property and the weak block property.

OBSERVATION 5. Suppose T is a (x,y)-tree with a (x,y)-numbering π. Let $T_1,T_2,\ldots,T_t$ denote the maximal subtrees in $F(\pi,T,1)$. Then the T_i are $(x-1,y_i+1)$-trees where

$$J(T_i) = (J(T) \cap V(T_i)) \mathbin{\bar{\bigcup}} (V(T_i) \cap P(\pi,T)),\ |J(T_i)| = y_i+1$$

and $\sum_{i=1}^{t} y_i = y$.

We define $f(x,y) = \max\{f(T)\colon T$ has an (x,y)-numbering$\}$.

It is easy to see that $f(x,y)$ is increasing in x and in y. We also write $f(x) = f(x,0)$.

OBSERVATION 6. $f(x,y) \leqslant 1+f(x-1,y+1)$.

Proof. This follows from Observation 5.

OBSERVATION 7. $f(x) \geqslant 1+f(x-1)$.

Proof. Let T be a tree with a $(x-1,0)$-numbering π and $f(T) = f(x-1,0)$. Consider a tree T' which is the union of 3 copies of T and a path P with three vertices adjacent to vertices of T. Obviously $f(T') \geqslant 1+f(T)$. T' is a $(x,0)$-tree since we can form a $(x,0)$-numbering π' on (a refinement of) T' so that for any vertex v in the ith copy of T we have $\pi'(v) = s_{a_i}k_a\pi(v)$ $a_i = i\cdot|V(T)|\lceil a/(a-1)\rceil$ and the vertices in P are numbered by a chain of numbers at most x apart. We then have $f(x) \geqslant f(T') \geqslant 1+f(x-1)$.

OBSERVATION 8. $f(x,1) \leqslant 1+f(x)$.

Proof. Suppose π is a $(x,1)$-numbering for a tree T and $u_0 = J(T)$. Let S consist of all edges $\{u,v\}$ of T such that $\pi(u) < \pi(u_0) < \pi(v)$. If $S = \emptyset$, then T is a $(x,0)$-tree and $f(T) \leqslant f(x)$. Suppose $S \neq \emptyset$. We now choose u_1,v_1,u_2,v_2 (not necessarily distinct) satisfying:

$$\begin{aligned}
\pi(u_1) &= \max\{\pi(u)\colon \ \{u,v\} \in E(T), \ \ \pi(u) < \pi(u_0) < \pi(v)\}\ , \\
\pi(v_1) &= \min\{\pi(v)\colon \ \{u_1,v\} \in E(T), \ \ \pi(u_0) < \pi(v)\}\ , \\
\pi(v_2) &= \min\{\pi(v)\colon \ \{u,v\} \in E(T), \ \ \pi(u) < \pi(u_0) < \pi(v)\}\ , \\
\pi(u_2) &= \max\{\pi(u)\colon \ \{u,v_2\} \in E(T)\colon \ \pi(u) < \pi(u_0)\}\ .
\end{aligned}$$

Let $\bar{P}$ denote a path containing u_1,u_2,v_1 and v_2. Any tree T' in the forest F' formed by removing the edges of $\bar{P}$ is a $(x,0)$-tree since for any edge $\{u,v\}$ in $S \cap E(T')$ the set $\{u'\colon \pi(u) < \pi(u') < \pi(v)\}$ must contain at least one vertex in $\{u_1,u_2,v_1,v_2\}-V(T')$. Thus by choosing a numbering with $\bar{P}$ (or its refinement) as the basic path we have

$$f(T) \leqslant 1+\max_{T'\in F'} f(T') \leqslant 1+f(x)\ .$$

OBSERVATION 9. $f(0,y) \leqslant y/2$.

Proof. Suppose a tree T has a $(0,y)$-numbering π. If v is a vertex in $V(T)-J(T)$ and $\{u,v\} \in E(T)$, then $|\pi(u)-\pi(v)| \leqslant |\{w \in J : \pi(u) < \pi(w) < \pi(v)\}| \leqslant |\pi(u)-\pi(v)| - 1$, which is impossible. Thus we can have at most y nontrivial vertices (vertices with degree $\geqslant 1$). By Lemma 4 we have $f(0,y) \leqslant y/2$.

OBSERVATION 10. Suppose π is a $(x,0)$-numbering for T. Suppose T' in $F(\pi,T,i)$ is a $(x-i,j)$-tree, $j \leqslant i$. Then the induced numbering π' of π on T' can be derived from a $(x-i+j,0)$-numbering.

Proof. For $1 \leqslant k \leqslant i$, let T_k be the maximal tree in $F(\pi,T,k)$ containing T'. From the proof of Observation 4 we know that $|\bigcup (P(\pi,T_k) \cap V(T'))| = j' \leqslant j$. Let $\bar{T}$ denote a forest which is the union of j paths and T' such that a vertex in the kth path coincides with the vertex in $P(\pi,T_k) \cap V(T')$ if $P(\pi,\pi_k) \cap V(T') \neq \emptyset$. We can extend $\pi/V(T')$ to $\bar{T}$ and obviously $\bar{T}$ has a $(x-i+j,0)$-numbering.

OBSERVATION 11. Suppose T has a (x,y)-numbering π, and π is derived from a $(x+y,0)$-numbering. Suppose $f(T) > f(x)+1$. Then $y \geqslant x+1$.

Proof. Clearly it holds for $x=1$. Suppose it is true for $x' < x$. Suppose $f(T) > f(x)+1$ and $y \leqslant x$. Since by Observations 6 and 7 $f(T) \leqslant f(x-1,y+1)+1$, and $f(x-1)+1 \leqslant f(x)$, we then have $y \geqslant x-1$. This implies $y = x-1$, or x. From Observation 4 a subtree in $F(\pi,T,1)$ is a $(x-1,y+1)$-tree. Let T_0 denote the maximal subtree in $F(\pi,T,1)$ with the maximum cutwidth.

If T_0 is a $(x-1,y-1)$-tree, by Lemma 5 and Observation 7 we have $1+f(T_0) \geqslant f(T) > 1+f(x) \geqslant 2+f(x-1)$. This implies $y \geqslant x+1$ which is impossible. Thus one of the subtrees is a $(x-1,x+1)$-tree or a $(x-1,x)$-tree, (denoted by T_0) and the rest are $(x-1,1)$-trees (with one exception of a $(x-1,2)$-tree by Observation 5). Clearly the vertex u of T_0 on the basic path $P(\pi,T)$ is in $J(T_0)$. Let $\bar{P}$ denote the path containing the largest number of different vertices in $J(T_0)$. We consider the following three possibilities.

CASE 1. $J(T_0)$ has three or more distinct vertices. Choose a numbering π of a refinement of T_0 so that $\bar{P}$ is the basic path. Suppose $|V(\bar{P}) \cap J(T_0)| \geqslant 3$. Since all trees in $F(\pi_0,T_0,1)$ are $(x-1,x-1)$-trees, we have $f(T) \leqslant 1+f(T_0) \leqslant 2+f(x-1,x-1) \leqslant 2+f(x-1) \leqslant 1+f(x)$, which is impossible. We may assume $V(P) \cap J(T_0) = \{v_1,v_2\}$. Again each subtree in F_0 can have at most $x-1$ vertices in J since the subtree contains v_i, $i = 1$ or 2, and does not contain any vertex in J. Thus we have

$$f(T) \leqslant 2+f(x-1,x-1) \leqslant 2+f(x-1) \leqslant f(x)+1 .$$

This is a contradiction. Therefore Case 1 cannot happen.

CASE 2. $J(T_0)$ has exactly one vertex i.e., $J(T_0)$ is a multi-set containing u, repeated y times. Let S denote the set of all ordered pairs (u',v') such that $\{u',v'\}$ is an edge and $\pi(u') \leqslant \pi(u) < \pi(v')$. If $S = \varnothing$, then T_0 is a $(x-1,0)$-tree and we have $f(T_0) \leqslant f(x-1)$. Thus $f(T) \leqslant 1+f(x-1) \leqslant 1+f(x)$, which is impossible. We may assume $S \neq \varnothing$. Let $(u',v') \in S$. Since π is derived from a $(x+y,0)$-numbering π', we know that the set $\{v\colon \pi'(u) < \pi'(v) \leqslant \pi'(u)+x+y\}$ contains at least $y+1$ vertices not in T_0 (one vertex on each basic path). Thus $\pi(v')-\pi(u) \leqslant x-1$. Similarly we can prove $\pi(u)-\pi(u') \leqslant x-1$. Therefore $\pi(v')-\pi(u') \leqslant 2(x-1)$. Thus T_0 is a $(x-1,x-1)$-tree and we have

$$f(T) \leqslant 2+f(x-1) \leqslant 1+f(x) .$$

Again this is a contradiction.

CASE 3. $J(T_0)$ has exactly two vertices, i.e. $J(T_0)$ consists of u, repeated i times and v, repeated $y-i$ times. If both i and $y-i$ are greater than one, the proof is similar to Case 1. If either i or $y-i$ is one, then the proof is similar to Case 2 and will be omitted.

Now we are ready to prove the main theorem.

THEOREM 4. *Suppose a tree T has topological bandwidth $b^*(T) = n$. Then $f(T) \leqslant n+\log_2 n+2$.*

Proof. We will prove by induction on n that $f(T) \leqslant n+\log_2(n-3)+2$ for a tree T with $b^*(T) = n$. It is true for $n \leqslant 4$ since $f(T) \leqslant n+f(0,n) \leqslant 3n/2$ by Observation 9. Let π denote the $(n,0)$-numbering of T. Then maximal subtrees in $F(\pi,T,i)$ are $(n-i,i)$-trees. Let T_i denote the maximal subtree in $F(\pi,T,i)$ with the largest cutwidth. Let z denote the largest integer satisfying

$$f(T_z) \leqslant f(n-z)+1 .$$

From Observation 8 we have $z \geqslant 1$. By definition we have $f(T_{z+1}) > 1+f(n-z-1)$. Using Observation 11 we have $z+1 \geqslant n-z-1$ which implies $z \geqslant n/2 - 1$. From Observation 5, we have

$$\begin{aligned}
f(T) &\leqslant z+f(T_z)\\
&\leqslant z+1+f(n-z) \quad \text{(by definition)}\\
&\leqslant z+1+(n-z)+\log_2(n-z)+2 \quad \text{(by induction)}\\
&\leqslant \frac{n}{2}+n-\frac{n}{2}+1+\log_2\Bigl(n-\frac{n}{2}+1-3\Bigr)+2 \quad \Bigl(\text{because } z \geqslant \frac{n}{2}-1\Bigr)\\
&\leqslant n+\log_2\Bigl(\frac{n}{2}-2\Bigr)+3\\
&\leqslant n+\log_2(n-4)+2\\
&\leqslant n+\log_2(n-3)+2 .
\end{aligned}$$

Thus we have shown that, if $b^*(T) = n$, then

$$f(T) \leqslant n+\log_2(n-3)+2 .$$

This completes the proof of Theorem 4.

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