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The Number of Different Distances Determined by n Points in the Plane

F. R. K. CHUNG

Bell Communications Research, Murray Hill, New Jersey 07974

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A classical problem in combinatorial geometry is that of determining the minimum number $f(n)$ of different distances determined by n points in the Euclidean plane. In 1952, L. Moser proved that $f(n) > n^{2/3}/(2\sqrt[3]{9}) - 1$ and this has remained for 30 years as the best lower bound known for $f(n)$. It is shown that $f(n) > cn^{5/7}$ for some fixed constant c.

I. INTRODUCTION

Suppose we have n distinct points in the Euclidean plane. There are $\binom{n}{2}$ distances determined by pairs of these n points. In 1946, Erdös [2] raised the question of finding the least number $f(n)$ of different distances determined by n points and proved that

$$\sqrt{n-1} - 1 < f(n) < cn/\sqrt{\log n}$$

where c is a fixed constant.

In 1952, Moser [4] improved the lower bound to

$$f(n) > n^{2/3}/(2\sqrt[3]{9}) - 1$$

and this has stood for 30 years as the best lower bound known for $f(n)$.

In this paper we will prove that

$$f(n) > cn^{5/7}$$

for a fixed constant c.

The upper bound $cn/\sqrt{\log n}$ for $f(n)$ was obtained by considering the points of a square lattice. Erdös conjectured that $f(n) > n^{1-\varepsilon}$ for any positive ε and n sufficiently large. This conjecture remains open.

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II. Preliminaries

In this section we will quote some known facts and prove several auxiliary lemmas. First we define some useful notation.

Supose S, S' are sets of distinct points in the plane and x is a point. Let $g(x, S)$ denote the number of different distances from x to points in S. Let $g(S, S')$ denote the number of different distances $d(u, v)$, for u in S and v in S'. Also let $D(S)$ denote the maximum distance between two points in S.

Lemma 1 [2, 4]. *Suppose x and y are two distinct points not in S. Then*

$$g(x, S)\, g(y, S) \geqslant |S|/2.$$

Lemma 2.

$$g(x, S) + g(y, S) \geqslant \sqrt{|S|}.$$

Proof. It follows from Lemma 1.

Lemma 3 [4]. *Let O be any fixed point and let x_1, x_2, x_3 be points satisfying the following (see Fig. 1):*

(i) $\angle x_i O x_j < 1°$ *for* $1 \leqslant i, j \leqslant 3$.

(ii) x_2 *and* x_3 *are on the same side of the through O and x_1.*

(iii) $r \leqslant d(x_i, O) < r + \frac{1}{2}$, $i = 1, 2, 3$, *for some positive value r.*

(iv) $d(x_1, x_2) = d(x_1, x_3)$.

Then $d(x_2, x_3) < 1$.

Lemma 4. *Let r and w be positive values and O be a fixed point. Let x_i, $i = 1, 2, 3, 4$, be points satisfying the following:*

(i) $\angle x_i O x_j < 1°$, *for* $1 \leqslant i, j \leqslant 4$.

(ii) *For* $i = 1, 2$, $r + w \leqslant d(O, x_i) < r + w + \frac{1}{2}$.
For $j = 3, 4$, $r \leqslant d(O, x_j) < r + \frac{1}{2}$.

(iii) $d(x_1, x_2) \geqslant 1$.

(iv) $d(x_1, x_j) = d(x_2, x_j)$, $j = 3, 4$.

Then $d(x_3, x_4) < 1$. *(See Fig. 2.)*

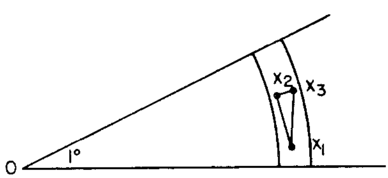


Figure 1

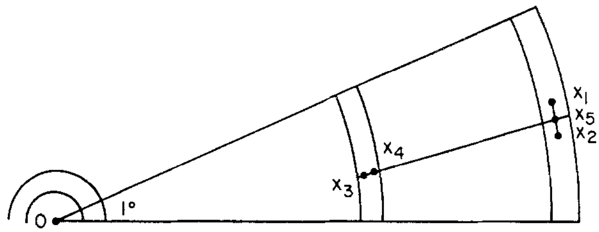


FIGURE 2

Proof. Let x_5 denote the midpoint of the segment $\overline{x_1x_2}$. Then x_3 and x_4 are on a line passing through x_5 and orthogonal to $\overline{x_1x_2}$. Furthermore the acute angle between the line x_5x_3 and the line x_3O is at most 40° since the angle determined by $\overline{x_1x_2}$ and Ox_1 is at least $\arccos\frac{1}{2} \geqslant 55°$. Therefore $d(x_3, x_4) < 1/(2\cos 40°) < 1$.

LEMMA 5. *Suppose r is an integer. Let X and Y denote two sets of points satisfying*:

(i) $r \leqslant d(x_i, O) < r + \frac{1}{2} \quad \forall x_i \in X$.

(ii) $w + r \leqslant d(y_i, O) < w + r + \frac{1}{2} \; \forall y_i \in Y$.

(iii) $\angle z_iOz_j < 1°$ *for* $z_i, z_j \in X \cup Y$.

(iv) $d(z_i, z_j) \geqslant 1 \; \forall z_i, z_j \in X \cup Y$.

(v) $|Y| \geqslant |X|$.

Then there are at least $|X|/4$ different distances between points in X and points in Y, i.e., $g(X, Y) \geqslant |X|/4$.

Proof. Let z denote the maximum number of different distances from a point in X to points in Y, i.e., $z = \operatorname{Max}\{g(x_i, Y): x_i \in X\}$.

Suppose $z < |X|/4$. For a fixed point x_i in X, we partition Y into sets $Y_{i1}, Y_{i2}, \ldots, Y_{iz'}, z' < z$, such that $d(x_i, y) = d(x_i, y')$ for y, y' in Y_{ij}. Therefore the number of equidistant pairs in Y from x_i is at least

$$\sum_{j=1}^{z'} \binom{|Y_{ij}|}{2} \geqslant z \binom{|Y|/z}{2}$$

where $\binom{x}{2}$ denotes the binomial coefficient function defined for all real x.

From Lemma 4 and (iv) we know that for any pair of vertices y_i and y_j in Y there is at most one point in X equidistant from y_i and y_j. Thus the total number of equidistant pairs in Y from some vertex in X is at least

$$|X| \, z \binom{|Y|/z}{2}.$$

Since the total number of pairs in Y is $\binom{|Y|}{2}$, we have (using the fact that $z < |X|/4 \leqslant |Y|/4$)

$$\frac{|X|\,|Y|^2}{4z} \leqslant \frac{|X|\,|Y|\,(|Y|-z)}{2z} \leqslant \binom{|Y|}{2}.$$

This implies $z \geqslant |X|/4$ which contradicts our assumption that $z < |X|/4$. This completes the proof of Lemma 5.

LEMMA 6. *Let O and O' be two fied points. Let x_1 and x_2 be two points satisfying the following (see Fig. 3):*

(i) $\angle x_i OO' \leqslant 1°$ *and* $\angle x_i O'O > 10°$ *for* $i = 1, 2$.

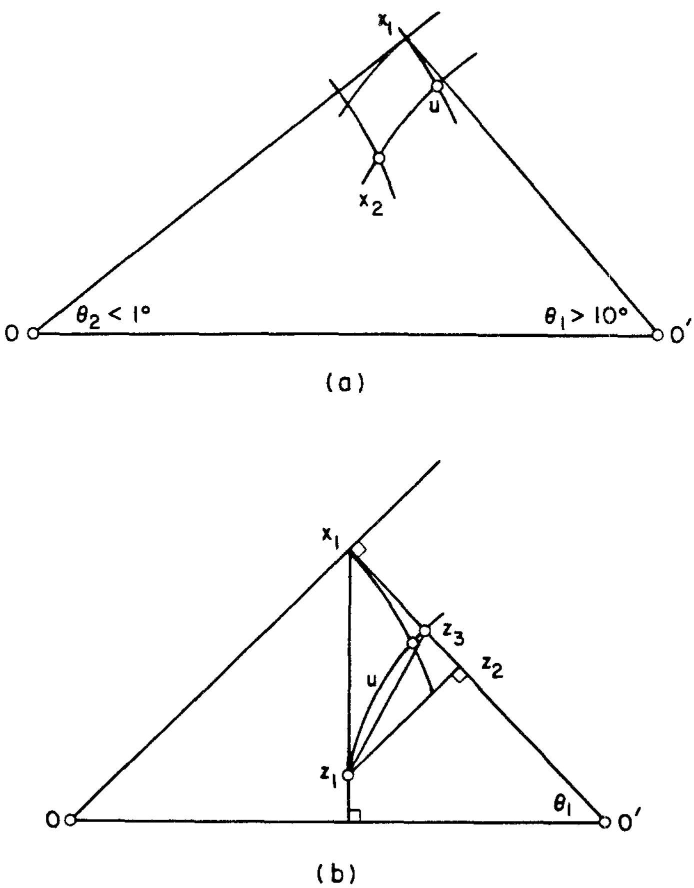


FIGURE 3

(ii) x_1 *and* x_2 *are on the same side of the line through* O *and* O'.

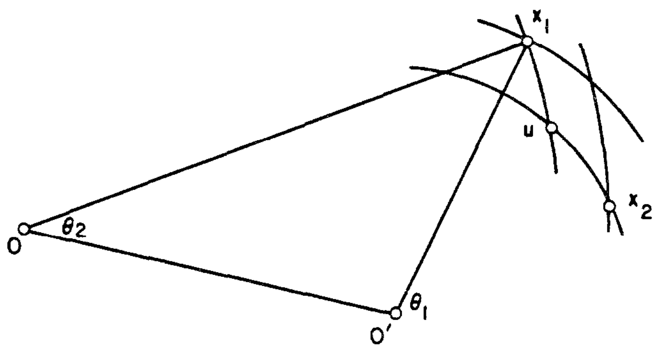
(iii) $|d(x_1, O) - d(x_2, O)| \leqslant 1/10$.

(iv) $|d(x_1, O') - d(x_2, O')| \leqslant 1/10$.

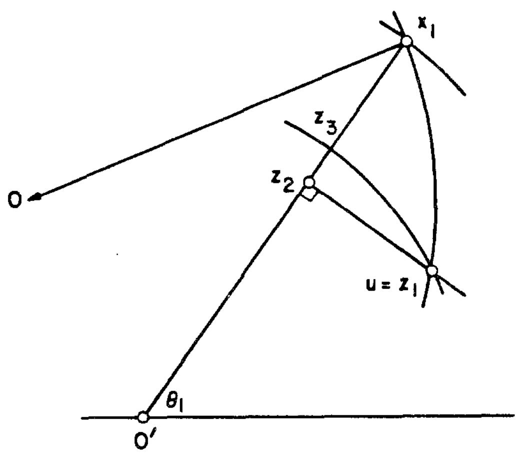
Then $d(x_1, x_2) < 1$.

Proof. Let u denote the point on the same side of OO' as x_1 and x_2 satisfying $d(x_1, O) = d(u, O)$ and $d(x_2, O') = d(u, O')$. It is not hard to check that (see Figs. 3(a)–(d))

$$d(x_1, u) \leqslant d(x_1, z_1) \leqslant d(x_1, z_2) \sec \angle O'x_1z_1$$
$$\leqslant d(x_1, z_2) \sec(89° - \theta_1)$$



(c)



(d)

FIGURE 3 *(continued)*

where θ_1 denotes the angle $\angle x_1 O'O$ and $\angle O'x_1 z_1 \geqslant 90° - \theta_1 - 1°$. Since $\angle z_1 z_3 z_2 \geqslant (180° - \theta_1)/2$ and

$$d(z_2, z_3) = d(z_1, z_2) \cot \angle z_1 z_3 z_2$$
$$\leqslant d(x_1, z_1) \cos(89° - \theta_1) \cot\left(\frac{180° - \theta_1}{2}\right),$$

we have

$$\frac{1}{10} \geqslant d(x_1, z_3) = d(x_1, z_2) - d(z_2, z_3)$$
$$\geqslant d(x_1, u)\left(\cos(89° - \theta_1) - \cos(89° - \theta_1) \cot\left(\frac{180° - \theta_1}{2}\right)\right)$$
$$\geqslant d(x, u) \cos 79°(1 - \cot 85°)$$
$$\geqslant d(x, u)/2.$$

This implies $d(x_1, u) < 1/5$.

Similarly it can be shown that

$$d(x_2, u) < 1/5.$$

Thus

$$d(x_1, x_2) < 1.$$

LEMMA 7. *For positive a_i, with $a_i \leqslant b$, we have*

$$\sum_i \sqrt{a_i} \geqslant \left(\sum a_i\right)\Big/\sqrt{b}.$$

Proof. This follows from the fact that $\sqrt{a_i} = a_i/\sqrt{a_i} \geqslant a_i/\sqrt{b}$.

III. ON THE LOWER BOUND OF $f(n)$

We will prove the following:

THEOREM.

$$f(n) \geqslant cn^{5/7}$$

for a fixed constant c.

Proof. Let X denote a set of n distinct points. Let c and c_i denote some constants to be specified later. Suppose there are fewer than $cn^{5/7}$ different

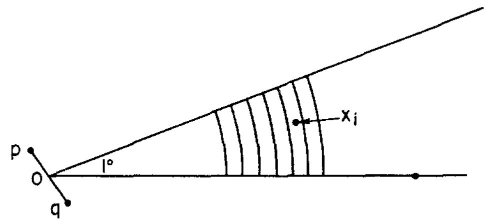


FIGURE 4

distances determined by points in X. Choose two points p and q in X such that $d(p, q)$ is the minimum distance determined by pairs of points in X. We may assume, without loss of generality that $d(p, q) = 1$. Let O denote the midpoint of the segment $\overline{pq}$. There exist at least $n/360$ points of X in some sector of 1° (see Fig. 4), i.e., there exists a set X' of $c_1 n$ points from X, where $c_1 \geqslant 1/400$ for $n > 20$, such that for x_i and x_j in X the angle $\angle x_i O x_j$ is less than 1°. Omit points not in X'. Now with center at O we construct circles of radii $i/10$, $i = 1, 2, \ldots$, cutting the sector into arc-shaped stripes. Now we partition the stripes into 11 classes, putting a stripe in class i if its inner radius is $j/10$ where $j \equiv i \pmod{11}$, $0 \leqslant i < 11$. At least one of the classes will contain at least $c_2 n$, $c_2 = c_1/11$, points. We will only deal with these points, called the set X'', and ignore the rest. Let A_i denote the set of points in the stripe with inner radius $i/10$, i.e., $A_i = \{x \in X'': i/10 \leqslant d(O, x) < (i+1)/10\}$, and set $a_i = |A_i|$. For $i \neq j$, the distances from a point u in A_i to p and q are different from the distances from a point v in A_j to p and q since for $x \in \{p, q\}$, $y \in A_i$, we have

$$\frac{i}{10} - \frac{1}{2} < d(x, y) \leqslant d(x, O) + d(O, y) \leqslant \frac{1}{2} + \frac{i+1}{10}$$
$$\leqslant \frac{i+11}{10} - \frac{1}{2}.$$

Let X''' denote the set of all points in X'' in stripes that contain more than $n^{4/7}$ points. Suppose there are more than $c_2 n/2$ points in $X'' - X'''$. Then we consider the numbers of different distances from p and q to points in $X'' - X'''$. Thus,

$$d(p, X'' - X''') + g(q, X'' - X''') \geqslant \sum_{A_i \subset X'' - X'''} (g(p, A_i) + g(q, A_i))$$
$$\geqslant \sum \sqrt{a_i} \qquad \text{(By Lemma 2).}$$

Since $\sum_{A_i \subset X''-X'''} a_i > c_2 n/2$ and $a_i < n^{4/7}$, we have from Lemma 7 that

$$g(p, W''-X''') + g(q, X'-X''') \geqslant c_2 n^{5/7}/2.$$

This implies that $g(\{p,q\}, X''-X''') \geqslant c_2 n^{5/7}/4 \geqslant cn^{5/7}$.

We only have to consider the case that X''' contains at least $c_3 n$, $c_3 = c_2/2$, points. If a stripe A_i contains $n^{5/7}$ points, then by Lemma 3 there are $n^{5/7}$ different distances determined by the points in A_i. We may assume $|A_i| < n^{5/7}$. Furthermore we will delete all points not in X'''. Now we partition the set of all stripes into "boxes" $B_1, B_2, ..., B_s$ as follows:

(i) For each i find the A_{j_i} with the least inner radius $r(A_{j_i})$ in $X''' - \bigcup_{k<i} B_k$.

(ii) Find the $A_{j_i'}$ with the least inner radius such that

$$\left| \bigcup \{A_k : r(A_{j_i}) \leqslant r(A_k) \leqslant r(A_{j_i'})\} \right| > n^{6/7}.$$

(iii) Set

$$B_i = \bigcup \{A_k : r(A_{j_i}) \leqslant r(A_k) \leqslant r(A_{j_i'})\}.$$

The width of B_i, denote by $w(B_i)$, is $r(A_{j_i'}) - r(A_{j_i})$ and the inner radius $r(B_i)$ of B_i is $r(A_{j_i})$.

It is easy to see that, for each i, $n^{6/7} < |B_i| < 2n^{6/7}$. Since each A_j has at most $n^{5/7}$ points and at least $n^{4/7}$ points, we have that the number s of boxes satisfies $c_3 n^{1/7}/2 < s < 2n^{2/7}$. We also need the following useful facts which will be proved later.

Claim 1. Suppose U is the union of $B'_{i_1}, ..., B'_{i_t}$ where B'_{i_j} is a subset of B_{i_j} with $|B'_{i_j}| \geqslant n^{6/7}/10$. Then there exists a subset $R = R(B'_{i_1}, B'_{i_2}, ..., B'_{i_j}) \subseteq U$ satisfying the following:

(i) R is a sector, i.e., there exists a point O' such that for any point v in R the acute angle determined by vO' and OO' is no more than 10°.

(ii) For each B'_{i_j}, $1 \leqslant j \leqslant t$, we have

$$|B'_{i_j} \cap R| > |B'_{i_j}|/2.$$

Claim 2. For every B_i we can find a part S_i of a stripe A_i in B_i satisfying the following:

(i) $|S_i| \geqslant n^{4/7}/10$.

(ii) Let $\bar{B}_i$ denote the set of $n^{6/7}/5$ points with the largest distances from O in B_i. Then for any point v in S_i and any point u in $R(\bar{B}_i)$ the acute angle between the line uv and vO is at most 11°.

(iii) Let $\bar{\bar{B}}_i$ denote the set of $n^{6/7}/5$ points with the smallest radii in B_i. Then for any point v in S_i and any point u in $R(\bar{\bar{B}}_i)$ the acute angle between the line uv and the line vO is at most 11°.

(iv) $D(S_i)$ is less than half of the width of B_i.

Let B_o denote the B_i with minimum width. From Lemma 5 we have

$$g(S_o, S_i) \geqslant n^{4/7}/40 \qquad \text{for each } i.$$

We need the following fact which will also be proved later.

Claim 3. Suppose $\bar{d} = d(u, v)$, where $u \in S_i$ and $v \in S_o$. Then there are at most 40 S_j's such that $\bar{d}$ is a distance between a point in S_o and a point in S_j. From Claim 3 we have

$$\begin{aligned} g(S_o, X) &\geqslant \frac{1}{40} \sum_{i=1}^{s} g(S_o, S_i) \\ &\geqslant \frac{1}{40} \cdot \frac{n^{4/7}}{40} \cdot \frac{c^3 n^{1/7}}{2} \\ &\geqslant c_4 n^{5/7} \\ &\geqslant cn^{5/7} \end{aligned}$$

which again contradicts the assumption that $g(X) < cn^{5/7}$ (by choosing c approximately 10^{-9}).

It remains to prove the claims. First we will prove Claim 1:

Proof of Claim 1. Let c and d denote the two points in U determining the smallest distance in U. Let O' denote the mid point of the segment $\overline{cd}$ (see Fig. 5). Let R denote the set of all points x in U with $\angle xO'O < 10°$. Let K denote the $B_i' - R$. Suppose $|K| > n^{6/7}/20$. We may assume half of the points in K are above the line OO'. Now we construct semi-circles of radius

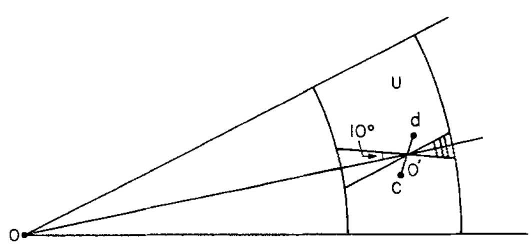


FIGURE 5

$(i/10)\, d(c, d)$, $i = 1, 2, \ldots$, centered at O', which cut B_i into stripes which we call $*$-stripes. Now we partition the $*$-stripes into 11 classes, putting a $*$-stripe in class i if its inner radius is $(j/10)\, d(c, d)$, where $j \equiv i \pmod{11}$, $0 \leqslant i < 11$. At least one of the classes will contain at least $n^{6/7}/440$ points. We will only deal with these $n^{6/7}/440$ points. Since we have $|A_i| > n^{4/7}$ and $|B_i| < 2n^{6/7}$, each $*$-stripe can contain at most $|B_i|/n^{4/7} \leqslant 2n^{2/7}$ points in K, based on Lemma 6 and the fact that $d(c, d) \geqslant 1$.

Let A_i^* denote a $*$-stripe with inner radius $(i/10)\, d(c, d)$. From Lemma 2 we have

$$g(c, A_i^*) + g(d, A_i^*) \geqslant \sqrt{|A_i^*|}.$$

Furthermore, for $i \neq j$, the distances from a point in A_i^* to c and d are different from the distances from a point in A_j^* to c and d since

$$\left(\frac{i}{10} - \frac{1}{2}\right) d(c, d) < d(x, y) \leqslant \left(\frac{i + 11}{10} - \frac{1}{2}\right) d(c, d)$$

for $x \in \{c, d\}$, $y \in A_i^*$. Thus we have

$$\begin{aligned} g(c, B_i') + g(d, B_i') &\geqslant \sum_i \sqrt{|A_i^*|} \\ &\geqslant \frac{n^{6/7}}{440 \cdot \sqrt{2n^{2/7}}} \\ &\geqslant \frac{n^{5/7}}{700} \geqslant cn^{5/7} \end{aligned}$$

since there are $n^{6/7}/440$ points in K and each A_i^* contains at most $2n^{2/7}$ points. Thus we have $g(X, X) \geqslant cn^{5/7}$ which contradicts our assumption. Therefore we conclude $|R \cap B_{i_j}'| \geqslant |B_{i_j}|/2$ for each j. This completes the proof of Claim 1.

Proof of Claim 2. Let K' denote the set of all points u in $B_i - \bar{B}_i - \bar{\bar{B}}_i$ with the property that for any point v in $R(\bar{B}_i)$ the (acute) angle between the line uv and uO is more than $11°$. Suppose $|K'| \geqslant n^{6/7}/10$. We consider $R(K' \cup R(\bar{B}_i))$. From Claim 1 we have $R(K' \cup R(\bar{B}_i)) \cap K' \neq \varnothing$ and $R(K' \cap R(\bar{B}_i)) \cap R(\bar{B}_i) \neq \varnothing$ since $R(\bar{B}_i) \geqslant |\bar{B}_i|/2 \geqslant n^{6/7}/10$. Now we choose u' in $R(K' \cup R(\bar{B}_i)) \cap K'$ and v' in $R(K' \cap R(\bar{B}_i)) \cap R(\bar{B}_i)$. The angle between the line $v'u'$ and $u'O$ is less than $11°$ since u' and v' are both in $R(K' \cap R(\bar{B}_i))$ and the angle $u'Ox$ is less than $1°$ for any point x in K'. This contradicts the assumption of u' being in K'. Thus we may assume $|K'| < n^{6/7}/10$.

Let K'' denote the set of all points u in $B_i - \bar{B}_i - \bar{\bar{B}}_i$ with the property that for any point v in $R(\bar{\bar{B}}_i)$ the angle between the line uv in uO is more than

11°. Similarly we can prove $|K''| < n^{6/7}/10$. Thus $B_i - \bar{B}_i - \bar{\bar{B}}_i - K' - K''$ contains at least $\frac{2}{5}n^{6/7}$ points. From Claim 1, $R(B_i - \bar{B}_i - \bar{\bar{B}}_i - K' - K'')$ contains at least $\frac{1}{5}n^{6/7}$ points. Since B_i contains at most $2n^{2/7}$ stripes, there is a subset S_i of a stripe contained in $R(B_i - \bar{B}_i - \bar{\bar{B}}_i - K' - K'')$ having $\frac{1}{10}n^{4/7}$ points. Furthermore, $D(S_i) \leqslant 2 \tan 10° w(B_i) \leqslant w(B_i)/2$.

Proof of Claim 3. Claim 3 will be proved by repeatedly using Claims 1 and 2. Suppose there exist u_j and v_j, $1 \leqslant j \leqslant 40$, $u_j \in S_{i_j}$ and $v_j \in S_o$ and

$$d(u_i, v_i) = d(u_j, v_j) = \bar{d}.$$

Obviously u_i, $1 \leqslant i \leqslant 40$, satisfies

$$\bar{d} - \frac{w}{2} \leqslant d(u_j, v_1) \leqslant \bar{d} + \frac{w}{2} \qquad \text{where} \quad w = w(B_o).$$

There are at least $20\, S_{i_j}$'s on the same side of $v_1 O$. Among these there ar $10\, S_{i_j}$'s with radii all larger than that of S_o or all smaller than that of S_o. Name these $S_{i_1}, ..., S_{i_{10}}$. We consider two cases.

Case 1. There are 5 j's, say $1 \leqslant j \leqslant 5$, such that the acute angle determined by $v_1 u_j$ and $u_j O$ is less than 45° (see Fig. 6(a)). We may assume $r(u_1) \leqslant r(u_2) \leqslant \cdots \leqslant r(u_5)$. Let Y denote the union of all stripes with radii between $r(u_1)$ and $r(u_5)$. From Claim 1 we know that $R(Y) \cap R(\bar{B}_{i_1}) \neq \varnothing$ and $R(Y) \cap R(\bar{\bar{B}}_{i_5}) \neq \varnothing$. The (acute) angle between $u_1 O$ and $u_1 u_5$ is at least 45° since the angle $v_1 u_1 O$ is at most 45° and $r(u_5) - r(u_1) \geqslant 3w$. By Claim 2, the angle between $u_1 O$ and $u_1 v_2$ for v_2 in $R(Y) \cap R(\bar{\bar{B}}_{i_1})$ is at most 11°. The angle between $v_2 O$ and $v_2 v_3$ for v_3 in $R(Y) \cap R(\bar{B}_{i_5})$ is no more than 20° since v_2 and v_3 are in $R(Y)$. The angle between $v_3 O$ and $v_3 u_5$ is at most 11°. Thus the acute angle between $u_1 O$ and $u_1 u_5$ is at most 45° since $\angle u_i O u_j$ is at most 1°. This yields a contradiction.

Case 2. The are 5 j's with $\angle v_1 u_j O > 45°$ (see Fig. 7). We may assume $r(u_1) < r(u_2) \cdots < r(u_5)$. Let Z denote the union of all stripes with radii between $r(v_1)$ and $r(u_5)$. Now we consider $R(Z)$. From Claim 1 we have $R(Z) \cap R(\bar{\bar{B}}_0) \neq \varnothing$ and $R(Z) \cap R(\bar{B}_{i_5}) \neq \varnothing$. The angle between $u_5 O$ and $u_5 v_2'$ for v_2' in $R(Z) \cap R(\bar{B}_{i_5})$ is at most 11°. The angle between $v_2' O$ and $v_2' v_3'$ for v_3' in $R(Z) \cap R(\bar{B}_0)$ is at most 11°. The angle between $v_3' O$ and $v_3' v_1$ is at most 20°. Thus the angle between $u_5 O$ and $u_5 v_1$ is at most 44° since $\angle u_i O u_j$ is at most 1°. This contradicts our assumption that $\angle u_1 u_5 O$ is greater than or equal to 45°. Thus Claim 3 is proved. The proof of our main theorem is now complete.

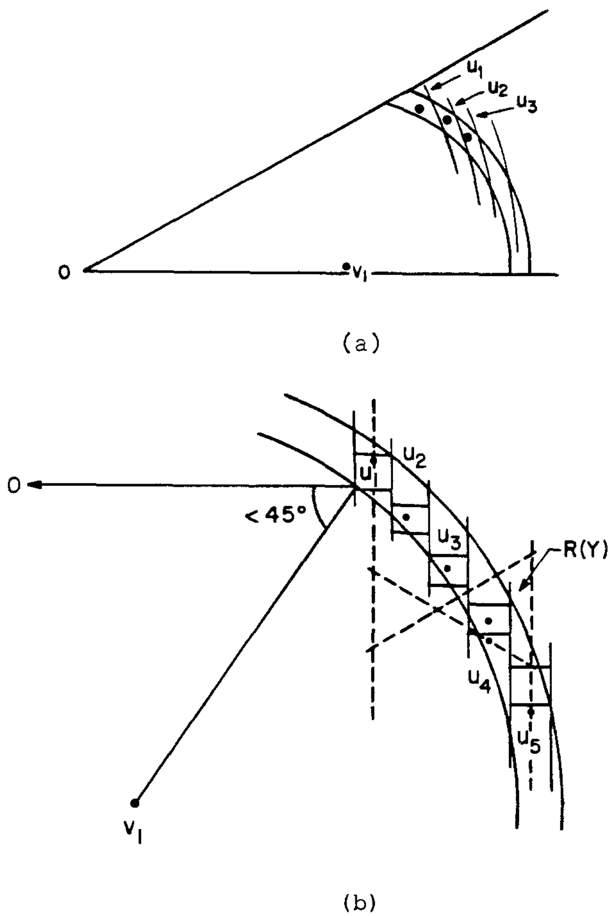


FIGURE 6

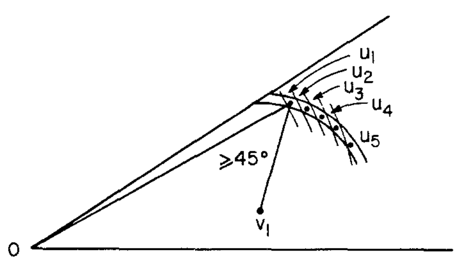


FIGURE 7

IV. Some Related Open Problems

Recently progress was made on another problem proposed by Erdös [2], of finding the maximum number $g(n)$ of pairs of n points determining the same distance. The value of $g(n)$ is related to the minimum number $f(n)$ of different distances on n points as follows:

$$g(n) \geqslant \binom{n}{2} \Big/ f(n).$$

Using the upper bound for $f(n)$, Erdös proved $g(n) > cn\sqrt{\log n}$. However, the above relation does not given very good estimates. In [2] Erdös proved

$$g(n) > n^{1+c/\log\log n}.$$

On the upper bound, Szemerédi [6] proved $g(n) = o(n^{3/2})$ and recently, Beck and Spencer [1] showed that

$$g(n) < n^{13/9}.$$

There are also many variations of enumerating the different distances determined by n points that satisfy certain conditions such as (1) all lie on a convex polygon, (2) no k of the points lie on a line; (3) every subset of l points determines at least m different distances. For a complete survey the reader is referred to [5].

Note added in proof. Recently, J. Beck proved $f(n) \geqslant n^{58/81}$ which can be improved by the author to $f(n) \geqslant n^{8/11}$. Szemerédi could further tighten the bound to $f(n) \geqslant n^{4/5}$.

References

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