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The Maximum Number of Edges in a 3-Graph Not Containing a Given Star

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Abstract. Suppose that $\mathscr{F}$ is a collection of 3-subsets of $\{1, 2, \ldots, n\}$ which does not contain a k-star (i.e., k 3-sets any two of which intersect in the same singleton). For $k \geq 3$ and $n \geq n_0(k)$, the collections having largest possible sizes are determined.

1. Introduction and Statement of the Results

Let $\mathscr{F}$ be a family of 3-subsets of $X = \{1, 2, \ldots, n\}$. Such a family is often called a 3-graph. A k-star is a collection of k distinct sets which intersect pairwise in the same one-element set. In the language of Erdös and Rado [8] a k-star is a strong Δ-system with kernel of size 1. One of the basic problems in extremal set theory is to find large families of sets which do not contain certain strong Δ-systems. This problem is not only interesting in its own right, but it also has applications in various problems arising in theoretical computer science.

Let $f(n, k)$ denote the maximum number of edges (or 3-sets) in a 3-graph without a k-star. For the case of $k = 2$, the value was determined exactly by Erdös and Sós [14]. Namely,

$$f(n, 2) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4 \\ n - 1 & \text{if } n \equiv 1 \pmod 4 \\ n - 2 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4 \end{cases}$$

The extremal families are basically the disjoint unions of 3-graphs consisting of all 3-subsets of a 4-element set. The situation gets more complicated for $k \geq 3$. Duke and Erdös [5] obtained lower and upper bounds for $f(n, k)$ which are linear in n for fixed k. Their bounds were improved in [11], where, $f(n, 3)$ was determined for $n \geq 54$. The bounds for $f(n, k)$ were further improved in [2], where, nearly best possible bounds were obtained.

The aim of this paper is to determine the exact value of $f(n, k)$ and the structure of the largest families which do not contain a k-star. Before we start analyzing such optimal families, we will first illustrate a few examples:

Example 1. Let G and H be two disjoint sets of X of size k. Consider a family of 3-sets of X $\mathscr{F}_k = \{F: (|F \cap G| \geq 2 \text{ and } |F \cap H| = \varnothing) \text{ or } (|F \cap H| \geq 2 \text{ and } F \cap G = \varnothing)\}$.

Suppose k is odd. Then it is easy to check that $\mathscr{F}_k$ contains no k-star. Furthermore, $|\mathscr{F}_k| = (n-2k)k(k-1) + 2\binom{k}{3}$.

Example 2. Let G_k be the (2-) graph with $2k-1$ vertices, called $x_1, \ldots, x_{k-1}$; $y_1, \ldots, y_{k-1}$ and z. The edge set of G_k consists of all the pairs (x_i, y_i), except for (x_i, y_i) with $2i > k$ together with the pairs (x_i, z), (y_i, z) with $2i > k$. It is easy to see that for k odd, G_k is regular of degree $k-1$ and for k even it has all degrees equal $k-1$ except for the degree of z, which is $k-2$.

Let $\mathscr{F}(G_k)$ consist of 3-subsets of X each of which either intersects the vertex set $V(G_k)$ of G_k in an edge of G_k or contains two edges of G_k. Again it is easy to check that $\mathscr{F}(G_k)$ contains no k-stars. If k is even, $\mathscr{F}(G_k)$, however, is not a maximum 3-graph containing no k-star. We can add to $\mathscr{F}(G_k)$ all the triples of the form $\{x_1, y_i, z\}$ with $1 \le i \le k/2$. The resulting graph, denoted by $\tilde{\mathscr{F}}(G_k)$ still contains no k-star.

Remark 1. It can be easily verified that, for k odd,

$$\begin{aligned}|\mathscr{F}(G_k)| &= (n-2k+1)\frac{(2k-1)(k-1)}{2} + (2k-1)\binom{k-1}{2} - 2\frac{(k-1)(k-3)}{4}\\ &= n\frac{(2k-1)(k-1)}{2} - (k-1)^2(k+2).\end{aligned}$$

A straightforward computation shows that $|\mathscr{F}(G_k)| \le |\mathscr{F}_k|$ if and only if $n \ge (4k^2 - 2k + 12)/3$.

The main results of this paper consist of the following two theorems. In the first theorem we determine exactly $f(n,k)$ for k odd and n large. We also show that the extremal graphs which achieve $f(n,k)$ are $\mathscr{F}_k$'s. In the second theorem we take care of the case of k being even by showing that $\tilde{\mathscr{F}}(G_k)$ are the extremal graphs which achieve $f(n,k)$.

Theorem 1.1. *Suppose that $k \ge 3$ is odd and $n > k(k-1)(5k+2)/2$. Then $f(n,k) = nk(k-1) + 2\binom{k}{3}$. Moreover, a 3-graph $\mathscr{F}$ has $f(n,k)$ edges and contains no k-star if and only if $\mathscr{F}$ is isomorphic to $\mathscr{F}_k$.*

Theorem 1.2. *Suppose that $k \ge 4$ is even and that $n \ge 2k^3 - 9k + 7$. Then $f(n,k) = |\tilde{\mathscr{F}}(G_k)| = (n-2k+1)\frac{(2k-1)(k-1)-1}{2} + (2k-2)\binom{k-1}{2} + \binom{k-2}{2} - \frac{(k-2)(k-4)}{2} + \frac{k}{2} = n\frac{k(2k-3)}{2} - \frac{1}{2}(2k^3 - 9k + 6)$. Moreover, a 3-graph $\mathscr{F}$ with $f(n,k)$ edges contains no k-star if and only if $\mathscr{F}$ is isomorphic to $\tilde{\mathscr{F}}(G_k)$.*

The proof of both theorems is based on properties of a weight function which is a simplified and improved version of the one used in [2]. In the case of even k there are further difficulties which lead to a new type of extremal graph problems (cf. section 5). These difficulties arise from the fact that many (2-) graphs with maximum degree $k-1$ contain no k independent edges and have the maximal number of $\frac{(2k-1)(k-1)-1}{2}$ edges. Each of these graphs can be used to construct a 3-graph without k-stars and the number of edges of such 3-graph is within a constant number of the optimum.

The paper is organized as follows. The weight function is introduced in Section 2. Section 3 establishes the main lemma concerning the weight function. The proof of Theorem 1.1 is given in Section 4. Section 5 deals with the minimum number of triangles in a graph with prescribed degree sequence. These results are used in Section 6 to prove Theorem 1.2. Section 7 gives a short overview of related problems.

2. A Weight Function

Suppose that $\mathscr{F}$ is a family of 3-element subsets of the n-set $V = V(H)$. Let P denote the set of all pairs of vertices in V. We define, for each $\{u, v\}$ in P, the pair frequency

$$z(u,v) = |\{w\colon \{u,v,w\} \in \mathscr{F}\}|.$$

We also define

$$A := \{\{u,v\} \in P\colon z(u,v) \geq 2k-1\}$$
$$B := \{\{u,v\} \in P\colon 2k-2 \geq z(u,v) \geq k\}$$
$$C := P - A - B$$

Now we define a weight function $w\colon \mathscr{F} \times P \to R$ which distributes weights to pairs within each triple in $\mathscr{F}$ according to the pair frequency.

For a fixed triple $T \in \mathscr{F}$ we denote by p_1, p_2, p_3 the three pairs in T with $z(p_1) \geq z(p_2) \geq z(p_3)$. The weight function w is defined as follows:

(i) If $p_1, p_2, p_3 \in A \cup B$ or $p_1, p_2, p_3 \in B \cup C$, then $w(T, p_i) = \frac{1}{3}$.

(ii) Suppose $p_1 \in A$, $p_3 \in C$. If $p_2 \in A \cup B$, then $w(T, p_1) = w(T, p_2) = \frac{1}{2}$, $w(T, p_3) = 0$. If $p_2 \in C$ then $w(T, p_1) = 1$, $w(T, p_2) = w(T, p_3) = 0$.

(iii) For convenience we set also $w(T, p) = 0$ for $p \notin T$.

Obviously, we have

$$\sum_{1 \leq i \leq 3} w(T, p_i) = 1 \tag{2.1}$$

and

$$\sum_{T \in \mathscr{F}} \sum_{p} w(T, p) = |\mathscr{F}| \tag{2.2}$$

For a vertex v, we define $N(v)$ to be the set of all pairs p with the property that the union of v and p is a triple in $\mathscr{F}$. Clearly $N(v)$ is a 2-graph.

Lemma 2.1. *Suppose that $\mathscr{F}$ does not contain a k-star. Let v be a fixed vertex and let $r_1(r_2, r_3)$ denote the number of vertices u with $\{u, v\} \in A(B, C)$ respectively.*

Then we have

(i) $r_1 \le k-1$ *and if* $r_1 = k-1$ *then* $r_2 = r_3 = 0$.

(ii) $r_1 + \frac{r_2}{2} \le k - \frac{1}{2}$.

Proof. (i) is obvious (cf. also [2], [10]). We only have to consider (ii).

Let G denote the induced subgraph of $N(v)$ on the set of all points u with $\{u,v\} \notin A$. If G contains $k-r_1$ vertex-disjoint edges, then we can find a k-star centered at v by considering these $k-r_1$ triples formed by these vertex-disjoint edges plus v and r_1 triples each containing exactly one of the points u' with $\{u',v\} \in A$. Therefore, we may assume that the maximum matching in G, say $E_1, \ldots, E_h$ satisfies $h \le k - r_1 - 1$. Define $Y = E_1 \cup \cdots \cup E_h$. We want to show that there is at most one vertex u with $\{u,v\} \in B$ which is not in Y. Suppose the contrary. Then there are two vertices u_1, u_2 not in Y and $\{u_1,v\}$, $\{u_2,v\} \in B$. For $E_j = \{x_1,x_2\}$ if $\{x_1,u_1\}$ is in G, then $\{x_2,u_2\}$ is not in G because replacing E_j by $\{x_1,u_1\}$ and $\{x_2,u_2\}$ would give a larger matching. Thus the number of edges from u_1, u_2 to Y is at most $2h$. Since $\{u_1,v\}$, $\{u_2,v\} \in B$, u_i, $i=1,2$, is adjacent to at least $k - r_1 \ge h+1$ points in G. Therefore, either u_1 or u_2 must be adjacent to some point not in Y which contradicts the assumption that the E_i's form a maximum matching. Therefore, we have $r_2 \le 2h+1 \le 2(k-r_1-1)+1$ as desired.

Lemma 2.2. *Suppose that $\mathscr{F}$ contains no k-star. Then for any two vertices u and v we have*

$$W = \sum_y w(\{y,u,v\},\{y,u\}) \le k-1. \tag{2.3}$$

Moreover, $W \le k - \frac{3}{2}$ *unless* $\{u,v\} \in C$, $z\{u,v\} = k-1$ *and for* $k-1$ *y's* $w(\{y,u,v\},\{y,u\}) = 1$ *holds.*

Proof. We consider three possibilities:

(i) Suppose $\{u,v\} \in A$.

By definition $w(\{y,u,v\},\{y,u\}) = 0$ unless $\{y,u\} \in A \cup B$ and the weight is at most 1/2. From Lemma 2.1 and $\{u,v\} \in A$ it follows that there are at most $2k-3$ choices of y contributing positive weight. Thus

$$W \le \tfrac{1}{2}(2k-3) = k - 3/2.$$

(ii) Suppose $\{u,v\} \in B$.

Now $z\{u,v\} \le 2k-2$ and each term in the summation is at most 1/3 except possibly terms with $\{y,u\} \in A$. However, by Lemma 2.1(i), this number is at most $k-2$. This gives an upper bound $\frac{1}{2}(k-2) + \frac{1}{3}k = k - 1 - \frac{k}{6}$.

(iii) Suppose $\{u,v\} \in C$.

Now $z\{u,v\} \le k-1$, which implies (2.3). If $z\{u,v\} \le k-2$, then $W \le k-2 < k - \frac{3}{2}$.

Similarly, $W \le k - \frac{3}{2}$ unless for $k-1$ y's one has $w(\{y,u,v\},\{y,u\}) = 1$ and thus $\{y,u\} \in A$ and $\{y,v\} \in C$ □

3. More Bounds on the Weight Function

A classical theorem of Berge [1] states that the size of a maximum matching, denoted by $\nu(G)$, in a graph G satisfies

$$\nu(G) = \min_{S} |S| + \frac{1}{2}(|V(G)| - |S| - c(G - S)) \tag{3.1}$$

where S ranges over all $S \subseteq V(G)$ and $c(G - S)$ denotes the number of odd components in the induced subgraph of G on $V(G) - S$.

Let now $\mathscr{H}$ be a 3-graph without a k-star and v be an arbitrary vertex of $\mathscr{H}$. Applying (3.1) to the neighborhood graph $G = N(v)$ we infer the existence of $S = S(v)$ satisfying

$$|S| + \frac{1}{2}(|V(G)| - |S| - c(G - S)) \leq k - 1.$$

Furthermore, let j_i denote the number of components of $G - S$ of size i. Then we have

$$\sum_{\substack{i \\ \text{even}}} \frac{i}{2} j_i + \sum_{\substack{i \\ \text{odd}}} \frac{(i-1)}{2} j_i + |S| \leq k - 1. \tag{3.2}$$

Lemma 3.1. *For every vertex v in a 3-graph $\mathscr{F}$ which does not contain a k-star, the following holds*:

$$W_v = \sum_{p \in N(v)} w(\{v\} \cup p, p) \leq k(k-1). \tag{3.3}$$

Moreover, $W_v \leq k(k-1) - \frac{2}{3}$ unless $G = N(v)$ is the disjoint union of two complete 2-graphs on k vertices and every edge of G is in A.

If k is even, then one has the stronger inequality

$$W_v \leq k\left(k - \frac{3}{2}\right). \tag{3.3'}$$

Moreover, $W_v \leq k(k - \frac{3}{2}) - \frac{1}{2}$ unless (a), (b) *and* (c) *hold.*

(a) *$G - S$ consists of isolated vertices and one connected component with $2k - 1 - 2|S|$ vertices and degree sequence $k - 1, \ldots, k - 1, k - 2$;*

(b) *the rest of G is the edge disjoint union of $|S|$ stars, each with degree $k - 1$; and*

(c) *every edge of G is in A, and all edges connecting v to G are in C.*

Proof. We break up the summation in (3.3) according to where the edge p is lying.

Let K be a component of $N(v) - S$. Then

$$\sum_{p \in E(C)} w(p \cup \{v\}, p) \leq \binom{|K|}{2} \tag{3.4}$$

$\left(\text{and} \leq \binom{|K|}{2} - 1 \text{ unless } K \text{ is a complete graph}\right)$.

If K has more than k vertices then (3.4) can be improved using Lemma 2.2:

$$\sum_{p\in E(C)} w(p\cup\{v\},p) = \frac{1}{2}\sum_{u\in K}\sum_{y\in K} w(\{y,u,v\},\{u,y\})$$

$$\le \frac{1}{2}|K|(k-1). \tag{3.5}$$

For edges p with at least one endpoint is S, we have

$$\sum_{p\in N(v),\,p\cap S\neq\varnothing} w(p\cup\{v\},p) \le \sum_{u\in S}\sum_{y} w(\{y,u,v\},\{u,y\})$$

$$\le |S|(k-1). \tag{3.6}$$

Suppose first that every component has no more than k vertices. Then (3.4), (3.6) and (3.2) imply, for k even,

$$\sum_{p\in N(v)} w(p\cup\{v\},p) \le \sum_{i<k} i\frac{i-1}{2}j_i + \frac{k(k-1)}{2}j_k + (k-1)|S|$$

$$\le (k-1)\Big(\sum_{i \text{ odd}} \frac{i-1}{2}j_i + \sum_{i \text{ even}} \frac{i}{2}j_i + |S|\Big) \le (k-1)^2.$$

For k odd, similarly we have

$$\sum_{p\in N(v)} w(p\cup\{v\},p) \le \sum_{i} i\frac{i-1}{2}j_i + (k-1)|S|$$

$$\le k\Big(\sum_{i \text{ odd}} \frac{i-1}{2}j_i + \sum_{i \text{ even}} \frac{i}{2}j_i + |S|\Big) \le k(k-1).$$

Moreover, one has $\le k(k-1)-1$ in either the following cases. $S\neq\varnothing$; $j_i\neq 0$ for some $i<k$. Suppose now that $W_v > k(k-1)-1$. Then $j_i = 0$ unless $i=k=$ odd and $S=\varnothing$. Therefore $N(v)$ has exactly two components of size k. In view of the remark in brackets at (3.4) both components are complete graphs of order k.

Since $\mathscr{F}$ contains no k-star, all pairs $\{u,v\}$, with u in the complete graphs, are in C. Thus for an edge $\{u_1,v_2\}$ of the complete graphs $w(\{v,u_1,u_2\},\{u_1,u_2\})\le\frac{1}{3}$ holds unless $\{u_1,u_2\}$ is in A. This concludes the proof of the lemma for the case of all components having no more than k vertices.

Next we consider the case when there is a component of size greater than k (by (3.2) one could not have two such components).

Let D denote this large component and suppose that it has d vertices. Again by (3.2) we have $d\le 2k-1$. Also, all the other components have size smaller than k.

Summing up (3.4), (3.5) and (3.6) gives

$$W_v \le (k-1)\Big(\sum_{i}\frac{i-1}{2}j_i + |S|\Big) + (k-1)\frac{d}{2}$$

$$\le (k-1)\Big(\sum_{i \text{ odd}}\frac{i-1}{2}j_i + \sum_{i \text{ even}}\frac{i}{2}j_i + |S|\Big) + \frac{k-1}{2} \le (k-1)\Big(k-\frac{1}{2}\Big). \tag{3.7}$$

For k odd the proof is complete.

Now we deal separately with the case k even. Let us define the partition $V(D) = L \cup M$, so that L consists of all vertices of D whose degree is at least k or who are adjacent to a vertex of degree at least k in $N(v)$.

Then for $u \in L$ one has $\sum w(\{y, u, v\}, \{y, u\}) \leq k - \frac{3}{2}$ by Lemma 2.2. Therefore the total weight distributed to edges of D is at most $\frac{1}{2}\left(|M|(k-1) + |L|\left(k - \frac{3}{2}\right)\right) = \frac{1}{2}d(k-1) - \frac{|L|}{4}$.

Now together with (3.4), (3.6) we get

$$W_v \leq (k-1)\left(\sum_{i \text{ odd}} \frac{i-1}{2} j_i + \sum_{\text{even}} \frac{i}{2} j_i + |S|\right) + \frac{k-1}{2} - \frac{|L|}{4}$$
$$\leq k\left(k - \frac{3}{2}\right) + \frac{1}{2} - \frac{|L|}{4}.$$

If $N(v)$ has one vertex of degree at least k, then $|L| \geq k + 1 \geq 4$. We obtain $W_v \leq k(k - \frac{3}{2}) - \frac{1}{2}$. We may assume that all vertices in D have degree at most $k - 1$.

The total weight distributed to edges in D cannot exceed the number of edges, i.e., it is at most

$$\lfloor |D|(k-1)/2 \rfloor = |D|(k-1)/2 - \tfrac{1}{2}.$$

Thus on the RHS of the first inequality in (3.7) we can write instead of $(k-1)d/2$ the above quantity, which is smaller by 1/2. This leads to

$$W_v \leq k\left(k - \frac{3}{2}\right).$$

Moreover, we have $W_v \leq k(k - \frac{3}{2}) - \frac{1}{2}$ unless D has degree sequence $k-1, k-1, \ldots, k-1, k-2$, and for every edge p in D $w(p \cup \{v\}, p) = 1$ and $p \in A$.

If $W_v > k(k - \frac{3}{2}) - \frac{1}{2}$ then equality must hold in (3.6) as well as $j_i = 0$ for $i < k$. Now Lemma 2.2 implies the rest of the statement.

This proves Lemma 3.1.

Remark 3.2. From the proof above we note that if b edges of D have $w(e \cup \{v\}, e) < 1$, then $W_v \leq k(k - \frac{3}{2}) - \frac{b}{2}$ holds.

4. The Odd Case

Suppose that for every $v \in V(\mathscr{F})$ one has

$$\sum_{p \in N(v)} w(\{v\} \cup p, p) \leq k(k-1) - \frac{2}{3} \tag{4.1}$$

Summing over v and using (2.2) yields

$$|\mathscr{F}| = \sum_{T \in \mathscr{F}} \sum_{p \subset T} w(T, p) = \sum_{v} \sum_{p \in N(v)} w(\{v\} \cup p, p) \leq nk(k-1) - 2n/3.$$

For $n > k(k-1)(5k+2)/2$, one has $nk(k-1) - 2n/3 < (n-2k)k(k-1) + 2\binom{k}{3}$. The statement of the theorem then follows.

Suppose now that for some $v \in V(\mathscr{F})$ (4.1) fails. By Lemma 3.1 $N(v)$ must be the disjoint union of two complete graphs, say with vertex sets S and R where $|S| = |R| = k$. Moreover, every pair p with $p \subset S$, or $p \subset R$ is in A.

Claim 4.1. *If $T \in \mathscr{F}$ then $|T \cap S| \neq 1$ and $|T \cap R| \neq 1$.*

Proof of the claim. Suppose without loss of generality that $T_0 \cap R = \{u\}$. Let $p_1, \ldots, p_{k-1}$ be the pairs in R containing u. Since p_i's, $1 \leq i \leq k-1$, are all in A, we can choose T_i, $1 \leq i \leq k-1$ such that $p_i \subset T_i$ and $T_i - \{u\}$, $0 \leq i \leq k-1$, are all pairwise disjoint. $T_0, T_1, \ldots, T_{k-1}$ form a star with center $\{u\}$, which is a contradiction. □

Claim 4.2. *If p is a pair with $|p \cap S| = 1$ or $|p \cap R| = 1$, then $p \in C$ holds.*

Proof. Suppose that $p \cap S = \{u\}$. If p has degree at least k, then one of the triples T containing p satisfies $T \cap S = \{u\}$, contradicting Claim 4.1. □

From Claims 4.1 and 4.2 it follows that for a vertex $u \in S (u \in R)$ the weight function $w(u \cup \{p\}, p)$ is zero unless $p \subset S$ ($p \subset R$, respectively). Even then, its value is only 1/3. Consequently: $\sum_p w(\{u\} \cup p, p) \leq \frac{1}{3}\binom{k-1}{2}$ holds.

Summing this over $u \in S \cup R$ gives

$$\sum_{u \in S \cup R} \sum_p w(\{u\} \cup p, p) \leq 2\binom{k}{3}. \tag{4.3}$$

Moreover, equality holds if and only if all triples in both S and R are edges of $\mathscr{F}$.

Combining (3.3) and (4.3) yields in analogy with (4.2):

$$|\mathscr{F}| = \sum_{u \in S \cup R} \sum_p w(\{u\} \cup p, p) + \sum_{u \in V - (S \cup R)} \sum_p w(\{u\} \cup p, p)$$
$$\leq 2\binom{k}{3} + (n-2k)k(k-1), \text{ as desired.}$$

If equality holds in the above expression, equality must hold in (3.3) for every $\tilde{v} \in V(\mathscr{F}) - (S \cup R)$. This implies that $N(\tilde{v})$ is the disjoint union of the two complete graphs $\binom{S}{2}$ and $\binom{R}{2}$.

Also, Claim 4.1 implies that the vertex set of the two complete graphs in $N(\tilde{v})$ cannot partition $S \cup R$ in a different way. Since equality in (4.3) implies $\binom{S}{3} \subset \mathscr{F}$ and $\binom{R}{3} \subset \mathscr{F}$, we obtained

$$\mathscr{F} = \left\{T \in \binom{V}{3} : |T \cap (S \cup R)| \geq 2 \text{ and } |T \cap S| \neq 1, |T \cap R| \neq 1\right\}.$$

This completes the proof of Theorem 1.1.

5. On the Minimal Number of Triangles in a Graph with Given Degree Sequence

Before we proceed to the case of k even, we examine an extremal problem for 2-graphs which is crucial to the proof in the next section. In this section all graphs will be 2-graphs. Let $k = 2t + 2 \geq 4$ and consider the graph G_k from the introduction with vertex set $\{x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}, z\}$ and edge set:

$$\{(x_i, y_j) \colon i \neq j\} \cup \{(x_i, y_i) \colon 1 \leq i \leq t + 1\} \cup \{(z, x_i), (z, y_i) \colon t + 2 \leq i < k\}.$$

It is easy to see that except for z, the graph G_k is bipartite and G_k contains $(t-1)t$ triangles. Let $\Delta(G)$ denote the number of triangles in G.

Theorem 5.1. *Suppose G is a graph on $4t + 3$ vertices and with degree sequence $2t + 1$, $2t + 1, \ldots, 2t + 1$, $2t$ then we have*

$$\Delta(G) \geq (t-1)t. \tag{5.1}$$

Moreover, if $\Delta(G) < t^2 - 1$ then G is isomorphic to G_k.

We remark that this leads to many interesting extremal problems which can be formulated as follows: For given degree sequence (instead of the number of edges) how many triangles (or other subgraphs) must G have? Such problems can be viewed as extension of the well-known theorem of Turán about the maximum number of edges in a graph which contains no small cliques.

Proof of Theorem 5.1. It is trivial for $t = 1$. Suppose $t \geq 2$ and that G contains fewer than $t^2 - 1$ triangles. Then the triangles cover at most $3t^2 - 6$ edges. Therefore there will be at least $\left\lfloor \frac{(4t+3)(2t+1)}{2} \right\rfloor - (3t^2 - 6) - 2t = t^2 + 3t + 7$ edges which are not contained in triangles and with both endpoints of degree $2t + 1$.

Let E_0 be the set of these, at least $t^2 + 3t + 7$ edges and for $e \in E_0$ let $\tau(e)$ be the number of triangles having one vertex in common with e. The number of edges in E_0 having one vertex in common with a fixed triangle is at most $|V(G)| - 3 = 4t$. Hence, $\sum_{e \in E_0} \tau(e) \leq 4t(t^2 - 2)$. By averaging we find $e = \{u, v\}$ so that

$$\tau(e) \leq \left\lfloor \frac{4t(t^2 - 2)}{t^2 + 3t + 7} \right\rfloor = m. \tag{5.2}$$

Let us set $A_o = \{v\} \cup N(u)$, $B_o = \{u\} \cup N(v)$. By definition $A_o \cap B_o = \varnothing$, $|A_o| = |B_o| = 2t + 1$. We know that there are altogether at most m edges inside A_o or B_o. The total degree of vertices in $A_o \cup B_o$ is $(4t+2)(2t+1)$ or $(4t+2)(2t+1) - 1$ according whether or not the vertex of degree $2t$ is in $A_o \cup B_o$. There is exactly one vertex w in $V(G) - A_o - B_o$.

Suppose, without loss of generality, that w is adjacent to no more vertices in A_o than in B_o. Let A_1 be the union of w and A_o. If A_1 contains a vertex v which is adjacent to fewer than t vertices in B_o then we can replace A_1 by $B_o \cup \{v\}$ and B_o by $A_1 - \{v\}$ so that the total number of edges inside A_1 or B_o decreases. Repeating this procedure we finally obtain a partition $A \cup B$, with $|A| = 2t + 2$, $|B| = 2t + 1$; every vertex in A is adjacent to at most t other vertices in A and there are at most $m + t$ edges inside A or inside B. There are two very similar cases:

Case (*a*). Suppose the vertex z of degree $2t$ is in A.

Let s be the number of edges inside B. Then degree considerations show that there are $s + t$ edges inside A.

Thus we have

$$2s \leq m. \tag{5.3}$$

Note that if $v \in A$, $v \neq z$, has degree d_v in A then it is not connected to exactly d_v vertices in B. For $v = z$, this number is $d_v + 1$. Consequently, the number of triangles in G with one edge inside A is at least

$$\begin{aligned}(s+t)(2t+1) &- \sum_{v \in (A - \{z\})} d_v^2 - d_z(d_z + 1) \\ &= (s+t)(2t+1) - d_z - \sum_{v \in A} d_v^2.\end{aligned} \tag{5.4}$$

Also, the number of triangles with one edge inside B is at least $s - (2t + 2 - 2s)$. Since $\sum_{v \in A} d_v = 2(s + t)$, it is easy to minimize (5.4) for a given s. In fact, it amounts to maximize $\sum d_v^2$, which is achieved by choosing the d_v as widespread as possible subject to the underlying constraints, i.e., $d_v \leq t$ and the d_v's form the degree sequence of a graph with $s + t$ edges.

For $s = 0$ the unique maximum is achieved by choosing $d_z = t$, $d_v = 1$ for t vertices and $d_v = 0$ otherwise. Then the value in (5.4) is bounded below by $t(2t + 1) - t - t^2 - t = t^2 - t$, and we obtain $G = G_k$. Note that if we choose $d_v = t$ for some other vertex instead of z, then the number of triangles comes out at least $t(2t + 1) - 1 - t^2 - t = t^2 - 1$. Similarly, if the maximum degree is at most $t - 1$ then we obtain at least $t(2t + 1) - (t - 1) - (t - 1)^2 - 2^2 - (t - 1) = t^2 + t - 3$ triangles. For $t = 2$ we get from (5.2) $s = 0$. We may assume $t \geq 3$. First, we will deal with the case $t = 3$ separately here. From (5.2) we derive $s \leq 1$. We only have to consider the case that $s = 1$. We note that the number of triangles with one edge inside B is at least 3. The value of (5.4) is bounded below by 7 (by choosing the d_v's 4, 2, 2, 1). The total number of triangles is $10 \geq t^2 - 1$.

Now it suffices to show that for $1 \leq s \leq m/2$ the value of (5.4) is at least $t^2 - 1$ for $t \geq 4$.

From (5.2), we can deduce $m < 4t$ and $s < 2t$. One can give explicitly the graph minimizing (5.4). Namely (5.4) is minimized by the graph on vertex set $\{0, 1, 2, \ldots, t\}$ and where $0, 1, \ldots, a - 1$ are connected to all other vertices and a is connected to 0, $1, \ldots, a - 1, a + 1, \ldots, a + b$ where at $a + b \leq t$ and $s + t = at + a + b - \binom{a+1}{2}$.

Therefore the following function $f(a, b)$ is a lower bound for (5.4):

$$\begin{aligned}f(a,b) = \left(at + a + b - \binom{a+1}{2}\right)(2t+1) - t - at^2 - (a+b)^2 \\ - b(a+1)^2 - (t - a - b)a^2\end{aligned}$$

We want to show that $f(a, b) \geq t^2 - 1$ for $1 \leq s \leq m/2$ $\left(\text{i.e. } t + 1 \leq at + a + b - \binom{a+1}{2} \leq t + \frac{m}{2}\right)$.

To prove this, the calculation is done by crude manipulation as follows:

(1) We note that the second derivative of $f(a,b)$ in b is -2, therefore the interior minimum is attained at $b=0$ or $t-a$. However the case for $b=t-a$ is equivalent to the case of a being one larger and $b=0$.

(2) From (5.2) we know that $a=1$ for $t=4, 5$ and $a \leq 3$ for $t \geq 6$. We only have to consider $f(a,0)$ for $a=2, 3, 4$ and the case of $a=1$ and $b=1$. Namely, it suffices to show that $f(1,1)$ and $f(2,0)$ are bounded below by t^2-1 for $t=4, 5$ and to show $f(1,1), f(2,0), f(3,0)$ and $f(4,0)$ are $\geq t^2-1$ for $t \geq 6$.

By straightforward calculation we get

$$\begin{aligned} f(1,1) &= t^2+t-5 \geq t^2-1 && \text{for } t \geq 4 \\ f(2,0) &= 2t^2-5t+3 \geq t^2-1 && \text{for } t \geq 4 \\ f(3,0) &= 3t^2-13t+15 \geq t^2-1 && \text{for } t \geq 5 \\ f(4,0) &= 4t^2-25t+42 \geq t^2-1 && \text{for } t \geq 6 \end{aligned}$$

This completes the proof of Case (a).

Case (b). Suppose z is in B. Let again s be the number of edges inside B. Then there are $s+t+1$ edges inside A and thus $2s+1 \leq m$. (5.4) is replaced by the lower bound

$$(s+t+1)(2t+1) - \sum_{v \in A} d_v^2.$$

Similar considerations as above give that this is always greater than $(t+1)(2t+1) - t^2 - 4 - t = t^2 + 2t - 3 \geq t^2 - 1$ □

By essentially the same, but technically somewhat easier proof one can determine the minimum number of triangles in a graph on $4t+1$ vertices and regular of degree $2t$.

Let F^* be the graph with vertex set $\{x_1, \ldots, x_{2t}, y_1, \ldots, y_{2t}, z\}$ and edges $\{x_i, y_j\}$ except for $i = j \leq t$ and $\{x_i, z\}, \{y_i, z\}, i \leq t$. It is easy to compute that $\Delta(F^*) = t^2 - t$.

Theorem 5.2. *Suppose that G^* is a graph of order $4t+1$ and regular of degree $2t$. Then $\Delta(G^*) \geq t^2 - t$. Furthermore equality holds if and only if G^* is isomorphic to F^*.* □

For the proof of Theorem 1.2 we need some more bounds on the number of triangles in graphs with given degree sequence.

Theorem 5.3. *Let $d > t \geq 1$ be integers, $d \leq 2t$ and suppose that G is a graph on $2d+1$ vertices with degree sequence $2t+1, \ldots, 2t+1, 2t$. Then* $\Delta(G) \geq \dfrac{(2d+1)(2t+1)}{6}(4t+1-2d) - \dfrac{2t+1}{6}$.

Proof. Let $\Omega(G)$ denote the number of triples (x,y,z) of vertices of G with (x,y), (x,z) edges and (y,z) a non-edge. If (y,z) is a non-edge, then it is contained in at most the minimum of the degree of y and z such triples.

This implies

$$\Omega(G) \le \left(\binom{2d+1}{2} - \frac{(2d+1)(2t+1)-1}{2}\right)(2t+1) - 2t.$$

On the other hand we have

$$2d\binom{2t+1}{2} + \binom{2t}{2} = \Omega(G) + 3\Delta(G).$$

Thus we infer

$$\begin{aligned} 3\Delta(G) &\ge \left((2d+1)\binom{2t+1}{2} - 2t\right) \\ &\quad - \left(\left(\binom{2d+1}{2} - \frac{(2d+1)(2t+1)-1}{2}\right)(2t+1) - 2t\right) \\ &= \frac{(2d+1)(2t+1)}{2}(4t+1-2d) - \frac{2t+1}{2} \qquad \square \end{aligned}$$

Note that already for $d = 2t$ we obtain $\Delta(G) \ge \dfrac{2t(2t+1)}{3}$, which is considerably more than the bound $t^2 - t$ from Theorem 5.1 for $d = 2t+1$.

6. The Even Case

Now we will prove Theorem 1.2 by considering the following three cases:

Case (a). For every vertex v one has $W_v \le k(k - \frac{3}{2}) - \frac{1}{2}$.

Summing over v gives

$$|\mathscr{F}| = \sum_v W_v \le nk\left(k - \frac{3}{2}\right) - \frac{n}{2} < |\mathscr{F}(G_k)| \quad \text{for } n \ge 2k^3 - 9k + 7.$$

Case (b). For some vertex v we have $W_v = k(k - \frac{3}{2})$ and $N(v) = G$ is a graph on $2k-1$ vertices with degree sequence $k-1, \ldots, k-1, k-2$.

Since Lemma 3.1 implies that every edge of G is in A, by Lemma 2.1(i) every triple $T \in \mathscr{F}$ with non-empty intersection with $V = V(G)$ satisfies one of the following conditions:

(i) $T \cap V$ is an edge of G.

(ii) $T \subset V$ and T contains at least 2 edges of G;

(iii) $T \subset V$, $z \in T$ and $T - \{z\}$ is an edge of G;

(iv) $T \cap V = \{z\}$.

Since $\mathscr{F}$ contains no k-star with center z, (iii) and (iv) cannot occur simultaneously. Also no two triangles in (iii) and (iv) intersect in only z. Let $\Delta = \Delta(G)$ be the number of triangles in G. Let there be q triples in (iii) and r in (iv), q and r can not both be nonzero.

By Lemma 3.1 all edges from v to V are in C. This, in particular, implies q, $r \le \max\{3, k-1\}$. Also for all other triples $F \in \mathscr{F}$ with $F \cap V \ne \varnothing$, $F \not\subset V$ and

$u \in F \cap V$ one has $w(F, F-\{u\}) = 0$. First we consider the subcase $G \simeq G_k$ and $r = 0$.

Now $|\mathscr{F}|$ can be expressed as

$$|\mathscr{F}| = \sum_{v \notin V} W_v + \left|\mathscr{F} \cap \binom{V}{3}\right| \leq (n-2k+1)k\left(k-\frac{3}{2}\right) + \sum_{v \in V} \binom{d_G(v)}{2} - 2\Delta + q.$$

Substituting the values $k-1, \ldots, k-1, k-2$ for $d_G(v)$, $\Delta = \frac{k-2}{2} \cdot \frac{k-4}{2}$ and using the fact that $q \leq \frac{k}{2}$, we have

$$|\mathscr{F}| \leq (n-2k+1)k\left(k-\frac{3}{2}\right) + (2k-2)\binom{k-1}{2} + \binom{k-2}{2} - \frac{(k-2)(k-4)}{2} + \frac{k}{2}$$
$$= |\mathscr{F}(G_k)|.$$

If equality holds, then for all vertices $v \notin V$ one has $W_v = k\left(k-\frac{3}{2}\right)$. This easily implies $N(v) = G_k$ for all v. Furthermore, there is essentially only one way to have $q = \frac{k}{2}$.

The subcase $G \neq G_k$, $r = 0$ is similar but somewhat simpler. Mainly, one notes $q \leq k-1$ and uses the proof of Theorem 5.1 that $2\Delta(G) - 2\Delta(G_k) \geq k-4$ and therefore $-2\Delta(G) + q < -2\Delta(G_k) + \frac{k}{2}$ holds for $k \geq 8$. The same follows for $k = 4$ from $\Delta(G) > 0 = \Delta(G_4)$, and for $k = 6$ unless $\Delta(G) = t^2 - 1 = 3$. In this case one can go through the proof of Theorem 5.1 to show that there is only one graph on 11 vertices, with the desired degree sequence and exactly 3 triangles and for this graph $q \leq 4$ holds.

In the only remaining subcase $G = G_k$, $q = 0$, one notes that for a triple $T = \{x, y, z\}$ in (iv) Lemma 3.1 implies $w(T, \{x, y\}) + W_x \leq k(k - \frac{3}{2}) + \frac{1}{2}$.

Thus we infer

$$|\mathscr{F}| = \sum_{v} W_v \leq (n-2k+1)k\left(k-\frac{3}{2}\right) + \frac{r}{2} + \left|\mathscr{F} \cap \binom{V}{3}\right|$$
$$\leq |\mathscr{F}(G_k)| + \frac{r}{2} - \frac{k}{2} < |\mathscr{F}(G_k)|.$$

Case (c). Suppose $W_v = k(k - \frac{3}{2})$ holds for some vertex v and $N(v)$ contains non-empty S as defined in Lemma 3.1.

Set $s = |S|$. We will use Remark 3.2 and Theorem 5.3. The rest of the proof will be quite similar to the preceding case. We will be somewhat sketchy in describing this proof. First note that if $T = \{x, y, z\}$ is a triple of which at least two pairs are in $N(v)$ then we can add T to $\mathscr{F}$ and the resulting 3-graph still has no k-star. If v is a vertex of D (as defined in the proof of Lemma 3.1) of degree $k-1$, then Lemma 3.1 implies that $w(T, T-\{v\}) = 0$ except possibly if $T \subset V(D)$ and at least two pairs of T are edges of D.

Similarly to the case (b) the "extra contribution" of the vertex z of D of degree $k-2$ is at most $k-1$. It suffices to prove that $f=\sum_{v\neq z} W_v$ is less than $|\mathscr{F}(G_k)|-(k-1)$.

Now for triples T with $T\cap S\neq\varnothing$ let us change the weight function – if necessary – so that W_u becomes zero for all $u\in S$. In fact if $u\in T\cap S$ then either one or two of the pairs of T through u are in $N(v)$ (and consequently are in A). We put all weight equally on these one or two pairs. One easily checks through the proofs, that the inequalities concerning the weight function remain valid.

Note also that if $u\in S$, y is a vertex of $N(v)$ which is connected to u, then in view of Remark 3.2 the other $k-2$ edges in $N(v)$, connected to u, reduce W_y by at least $\dfrac{k-2}{2}$.

These losses add up to at least $(k-1)\dfrac{k-2}{2}=\dbinom{k-1}{2}$.

Now we can write

$$f=\sum_{v\neq z} W_v \le (n-2k+1+s)k\left(k-\frac{3}{2}\right)-\binom{k-1}{2}+\left|\mathscr{F}\cap\binom{V(D)-z}{3}\right|$$

$$\le (n-2k+1+s)k\left(k-\frac{3}{2}\right)+(2k-3-2s)\binom{k-1}{2}+\binom{k-2}{2}-2\Delta(D).$$

Or, equivalently:

$$f\le|\mathscr{F}(G_k)|+s\left(k\left(k-\frac{3}{2}\right)-2\binom{k-1}{2}\right)-\binom{k-1}{2}-\frac{k}{2}-2(\Delta(D)-\Delta(G_k)).$$

Note that $k\left(k-\dfrac{3}{2}\right)-2\dbinom{k-1}{2}=\dfrac{3}{2}k-2.$

By applying Theorem 5.3 we obtain

$$f<|\mathscr{F}(G_k)|-k+\left(s\left(\frac{3}{2}k-2\right)+\frac{k}{2}-\binom{k-1}{2}-\frac{(2k-1-2s)(k-1)(2s-1)}{3}\right.$$
$$\left.+\frac{k-1}{3}+\frac{k-2}{2}\cdot\frac{k-4}{2}\right).$$

The term in brackets is a polynomial of degree 2 in s, which is negative for $k\ge 4$ at $s=1$ and $s=\dfrac{k-2}{2}$, thus it is negative for $1\le s\le(k-2)/2$ yielding the desired inequality:

$$f\le|\mathscr{F}(G_k)|-k \qquad \square$$

7. Related Problems

A general problem first considered by Duke and Erdös [5] is the following:

Let r, k, t be integers satisfying $0\le t<r$, $r\ge 3$, $k\ge 2$. Denote by $f(n,r,k,t)$ the maximum size of a family $\mathscr{F}\subset\dbinom{V}{r}$, $|V|=n$ such that $\mathscr{F}$ contains no *sunflower*

of type (k, t), i.e., there are no $F_1, \ldots, F_k \in \mathscr{F}$ so that for some t-element set T and all $1 \leq i < j \leq k$ one has $F_i \cap F_j = T$.

In fact, Erdös and Rado [8] have already introduced the function $\psi(r, k)$ which is the maximum size of a family $\mathscr{F}$ of r-element sets such that $\mathscr{F}$ contains no sunflower of size k, i.e., no $F_1, \ldots, F_k$ have pairwise the same intersection.

It is trivial that $\psi(r, 2) = 1$. In [8] it was proved that $\psi(r, k) \leq r!(k-1)^r$ and Erdös [7] proposes the problem of deciding whether $\psi(r, 3) < c^r$ holds for some absolute constant c.

Suppose now that $\mathscr{H}$ is a collection of $\psi(t+1, k)$ subsets of size $t+1$ containing no sunflower of size k. Let V be an n-element set containing $V(\mathscr{H})$ and define $\mathscr{F}(\mathscr{H}) = \left\{F \in \binom{V}{r} : F \cap V(\mathscr{H}) \in \mathscr{H}\right\}$. It is easy to check that $\mathscr{F}(\mathscr{H})$ contains no sunflower of type (k, t) and

$$|\mathscr{F}(\mathscr{H})| = (\psi(t+1, k) - o(1))\binom{n-t-1}{r-t-1}$$

This construction is believed to be essentially best possible for $r \geq 2t+1$. The following results confirmed several special cases of this conjecture.

$k = 2, t = 0$:	Erdös-Ko-Rado Theorem [9]
$k \geq 3, t = 0$:	Erdös [6]
$k = 2, r = 3, t = 1$	Erdös and Sós [14]
$k = 2, r > 2t + 1$	Frankl and Füredi [12]
$k \geq 3, r > 2t + 2$	Frankl and Füredi [13]

In this paper we gave the complete solution for the case $r = 3$, $t = 1$, $n > n_o(k)$. In [12] $f(n, r, k, t) = O(n^t)$ is proved for $r \leq 2t + 1$ and for all k.

Another general direction is the problem of unavoidable hypergraphs. Namely, for given n and m what is the maximum number $g(n, m, r)$ of edges in a r-graph G which is contained in every r-graph on n vertices and m edges. Various results concerning the cases $r = 2$, 3 can be found in [3] and [4]. The maximum unavoidable r-graphs are often not sunflowers but combinations of sunflowers of different types. Results on unavoidable sunflowers are often very useful in studying unavoidable r-graphs in general.

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