probability that there are $a-kd^2$ vertices in A not in any $N(S_i)$ is at most

$$\binom{a}{kd^2}(1-p^d)^{k(a-kd^2)} \leqslant 2^a e^{-p^d ka/2}.$$

Since there are at most b^{dk} choices for S_i, $1 \leqslant i \leqslant k$, the probability for a bipartite graph to be 'bad' is at most

$$b^{dk} \cdot 2^a e^{-p^d ka/2} < (a \log a)^{p^{-1}k} \cdot 2^a e^{-p^d ka/2}$$
$$< (a \log a)^{a \log\log a/\log a} 2^a e^{-a^2 p^{d-2}/4} < 1'$$

Therefore the required bipartite graph exists as claimed.

Claim 2. *Given positive integers a and b where* $a < b < a \log a$ *and* $\log\log\log a \geqslant 1$, *there is a bipartite graph H with vertex set* $A \cup B$ *where* $|A| = a$ *and* $|B| = b$ *satisfying the following conditions:*

(i) *H has no more than abp edges where* $p = \log\log a/\log a$.

(ii) *Let H′ be a bipartite graph with vertex set* $X \cup Y$ *where* $|X| \leqslant \frac{1}{2}a$, $|Y| = b$ *and maximum degree* p^{-1}. *Then H′ can be embedded in H in the* strong sense, *i.e. any one-to-one map* $\lambda: Y \to B$ *can be extended to* $\bar{\lambda}: X \cup Y \to A \cup B$ *such that* $\bar{\lambda}(u)$ *is adjacent to* $\bar{\lambda}(v)$ *in H if u is adjacent to v in H′.*

Proof. We take H to be the graph in Claim 1. The mapping λ will be extended to $\bar{\lambda}: X \cup Y \to A \cup B$ in the following way:

For a vertex x in X, we define

$$S(x) = \{b \in B : b = \lambda(y) \text{ and } y \text{ is adjacent to } x\},$$

$$M(x) = N(S(x)) = \{v \in A : v \text{ is adjacent to all vertices in } S(x)\}.$$

The existence of $\bar{\lambda}$ is equivalent to a system of distinct representatives for $\{M(x)\}_{x \in X}$.

It suffices to show that for any set $X' \subseteq X$ we have

$$\left|\bigcup_{x \in X'} M(x)\right| \geqslant |X'|.$$

This is clearly true for $|X'| \leqslant (\log a/\log\log a)^2$ by property (ii) of H.

Now suppose $|X'| > (\log a/\log\log a)^2$. Since H' is of bounded degree $d = \log a/\log\log a$, for each x there are at most d^2 vertices x' in X with $S(x) \cap S(x') \neq \emptyset$. Thus there is a subset X'' of X where $|X''| \geqslant |X'|/d^2$ such that all $S(x)$, $x \in X''$, are mutually disjoint. Therefore,

$$\left|\bigcup_{x \in X'} M(x)\right| \geqslant \left|\bigcup_{x \in X''} M(x)\right| \geqslant \frac{|X'|p^{-2}}{d^2} \geqslant |X'|.$$

This completes the proof of Claim 2.

Claim 3. *There exists a graph* $\bar{H}$ *with* $4n^2 \log\log n/\log n$ *edges which contains all graphs with n vertices and degree at most* $\log n/\log\log n = d$.