EXPONENTIAL SUMS

ROSS M. RICHARDSON

We present here a short primer on various exponential sums that occur in harmonic analysis.

1. Incomplete sums

Let $A = \{x_1, \dots, x_N\} \subset \mathbb{R}$ denote a finite subset. An *exponential sum* is a sum of the form

$$\sum_{x \in A} e^{2\pi i x} = \sum_{x \in A} e(x)$$

where we in introduce the standard notation $e(x) = \exp(2\pi i x)$.

Question 1.1. What is the magnitude of $|\sum_{x \in S} e(x)|$?

By the trivial bound, we find that $|\sum_{x \in S} e(x)| \leq N$, with equality whenever the terms are all equal. One can of course make the sum much smaller–simply sum the Nth roots of unity. In between these two extremes–completely in phase and maximally out of phase–we should consider what we expect for a generic or random sum.

Example 1.1. Let x_i be uniform independent variables on $[0, 1]$. Then

$$\begin{aligned}\mathbb{E}[|\sum_{i=1}^{N} e(x_i)|^2] &= \mathbb{E}[\sum_{1 \leq i,j \leq N} e(x_i - x_j)] \\ &= N + \mathbb{E}[\sum_{i \neq j} e(x_i - x_j)] = N.\end{aligned}$$

so $\mathrm{Var}(|\sum_{i=1}^{N} e(x_i)|) = O(N)$. Further, standard concentration inequalities for the one dimensional walk give us a bound $\mathbb{E}[|\sum_{i=1}^{N} e(x_i)|] = O(\sqrt{N})$, by applying the inequalities coordinate-wise.

Hence, we expect random-like sequences to have $O(\sqrt{N})$ bound. Hence, $O(N)$ and $O(\sqrt{N})$ serve as guides for the degree to which points are well-distributed in $[0, 1]$.

Let us now define an *incomplete Weyl sum* for a function $f : \mathbb{N} \to \mathbb{R}$ as

$$S_f(N) = \sum_{1 \leq n \leq N} e(f(n))$$

and in general

$$S_f(M, N) = \sum_{M < n \leq M+N} e(f(n)) = \sum_{1 \leq n \leq N} e(f(n + M)).$$

Remark 1.1. Let us note that the notion of being well-distributed can be given a formal defintion in the following way. We say that a sequence $x_1, x_2, \ldots$ is *equidistributed* on $[0,1]$ if for any $0 \leq a \leq b \leq 1$ we have

$$\lim_{n\to\infty} \frac{|\{i \leq n \mid a \leq x_i \leq b\}|}{n} = b - a.$$

It is not too difficult to show that for any Riemann integrable function $f : [0,1] \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \int_0^1 f(x)dx,$$

for any equidistributed sequence $\{x_i\}_{i=1}^{\infty}$.

Weyl showed that a sequence is equidistributed on $[0,1]$ if and only if for each integer $l \neq 0$ we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} e(lx_i) = 0,$$

i.e. the corresponding Weyl sum has a non-trivial estimate.

Now let us look at some special cases of Weyl sums.

Example 1.2. Set $f(x) = \alpha x$. Thus we have

$$\sum_{1\leq n\leq N} e(\alpha n) = \frac{\sin(\pi\alpha N)}{\sin(\pi\alpha)} e(\frac{\alpha}{2}(N+1)).$$

Thus, $|S_f(N)| \leq \min(N, \frac{1}{|\sin \pi\alpha}|)$. If we denote by $\|x\|$ the distance from x to the nearest integer, then we have the elementary inequality $|\sin \pi x| \geq 2\|x\|$. Hence, $|S_f(N) \leq \min(N, \frac{1}{2\|\alpha\|})$. As we would suspect, the best bound arises when $\|\alpha\|$ is near $1/2$.

Example 1.3. Now let $f(x) = \alpha x^2 + \beta x$. The Weyl sum in this case is also referred to as a Gauß sum, and in particular if a and b are integers and $(2a, N) = 1$ we have the equality (!)

$$\left|\sum_{1\leq n\leq N} e(\frac{an^2+bn}{N})\right| = \sqrt{N}.$$

This is the classical Gauß sum, and in particular is also a complete exponential sum. (GOOD EXERCISE)

Example 1.4. If we additionally know something about the Diophantine nature of the leading coefficient α, then we can say something better. Specifically, if

$$\left|2\alpha - \frac{a}{q}\right|$$

with $(a,q) = 1$ and $1 \leq q \leq 2N$, then we have

$$|S_f(N)| \leq 2Nq^{-1/2} + q^{1/2}\log q.$$

(DETAILS?)

For arbitrary polynomials f of degree k, Weyl discovered a useful trick to reduce to the case of degree one, at which point example 1.3 becomes relevent. Specifically, we can write

$$|S_f(N)|^2 = \sum_n \sum_m e(f(m) - f(n))$$
$$= \sum_{|l|<N} \sum_{\substack{1\le n\le N \\ 1\le l+n\le N}} e(f(l+n) - f(n)).$$

We observe that the inner sum is itself a Weyl sum of a polynomial, this time with degree $k-1$. One can thus recursively apply this trick until one obtains sums for polynomials of degree one. At this point, we may apply the bound of exmaple 1.3 and obtain for a polynomial $f(x) = \alpha x^k + \dots, \quad k \ge 1$,

$$|S_f(N)| \le 2N \left(N^{-k} \sum_{-N<l_1,\dots,l_{k-1}<N} \cdots \sum \min(N, \|\alpha k! l_1 \cdots l_{k-1}\|^{-1} \right)^{2^{1-k}} .$$

From the quadratic case, one might suppose that Diophantine information about α might give sharper bounds. We thus obtain Weyl's inequality.

Theorem 1.1 (Weyl)**.** *Let M, N, a, q be integers such that $(a,q) = 1$ and $q > 0$. If f is a real polynomial of degree $k \ge 1$ with leading coefficient α such that $|\alpha - \frac{a}{q}| \le tq^{-2}$ for some $t \ge 1$ then for any $\epsilon > 0$ we have*

$$\sum_{x=M+1}^{M+N} e(f(x)) = O\left(N^{1+\epsilon} \left(\frac{t}{q} + \frac{1}{N} + \frac{t}{N^{k-1}} + \frac{q}{N^k} \right)^{2^{1-k}} \right).$$

We observe that the bound is nontrivial when q is small compared with N^k. Gowers provides an intersting application of Weyl's inequality, and in fact has recourse to examine the implied constant in the big-oh above.

Weyl's inequality relies on f both being a polynomial and knowledge about the leading coefficient. Other natural conditions have been investigated,most notably the method of van der Corput. By estimating the given sums with integrals, one may arrive at exponential sum bounds that depend on knowledge of various derivatives. These methods, for instance, have been very sucessuly applied to the Riemann zeta function.

2. Complete sums

Complete sums typically refer to exponential sums over finite fields. In what follows we denote by $\mathbb{F} = \mathbb{F}_q$ a field of prime power order.

We shall principally be concerned with characters. The map $\psi : \mathbb{F} \to \mathbb{C}^*$ which is a homomorphism from the additive group of $\mathbb{F}$ we call an *additive character*. Similarly, a homomorphism $\phi : \mathbb{F}^* \to \mathbb{C}^*$ from the multiplicative group of F to $\mathbb{C}^*$ is a *multiplicative character*. We can extend the definition of any multiplicative character to all of $\mathbb{F}$ by setting $\phi(0) = 0$.

Example 2.1. Additive characters are fairly lowbrow. For q a prime, the functions $k \mapsto e(ak/q)$ are indeed all the characters for $a = 0, \dots, q-1$. For vector spaces $\mathbb{F}^k$ over a field $\mathbb{F}$, the characters are all of the form $x \mapsto e((x \cdot \xi)/q)$ for $\xi \in \mathbb{F}^k$.

Multiplicative characters are much more interesting, and form the hart of much of number theory. We mention the most basic nontrivial such character, the quadratic residue character χ', which is one on all quadratic residues modulo q and -1 otherwise.

Exponential sums for characters are fairly well understood. In particular, we have the character orthogonality relations

$$\sum_{x \in \mathbb{F}_q} \psi(x) = 0$$

if ψ is a nontrivial additive character. Similarly,

$$\sum_{x \in \mathbb{F}_q} \chi(x) = 0$$

if χ is a nontrivial multiplicative character. The cases where ψ and χ are trivial is left as an (easy) exercise.

There are a number of exponential sums which can be investigated based on additive and multiplicative characters. For $f \in \mathbb{F}[x]$ or $f \in \mathbb{F}(x)$, and ψ a nontrivial additive character, we can sum

$$S_f(\psi) = \sum_{x \in \mathbb{F}} \psi(f(x)).$$

If $f \in \mathbb{F}(x)$ it is understood that the sum is taken only at non-poles.

Similarly, for $g \in \mathbb{F}[x]$ or $g \in \mathbb{F}(x)$ we can form the sum

$$S_{f,g}(\psi, \chi) = \sum_{x \in \mathbb{F}} \psi(f(x))\chi(g(x)).$$

It is common to say the the exponential sum $S_f(\psi)$ is *twisted* by the multiplicative character χ in the above sum.

We now arrive at the two most commonly occuring sums in the theory, which are special cases of the above two. Here, ψ, ϕ are additive characters and χ is a multiplicative character.

$$G(\chi, \psi) = \sum_{x \in \mathbb{F}} \chi(x)\psi(x) \qquad \text{Gauß sum}$$

$$S(\psi, \phi) = -\sum_{x \in \mathbb{F}^*} \psi(x)\phi(x^{-1}) \qquad \text{Kloosterman sum}$$

We mention that another type of Kloosterman sum (the *classical* Kloosterman sum) exists, namely

$$S(a, b; c) = \sum_{x(mod c)}^{*} e\left(\frac{ax + bx^{-1}}{c}\right)$$

where here $a, b, c \geq 1$ are integers nnad $\sum^*$ denotes summation over $x < c$ and relatively prime to c. Note that here we make no requirement about summing over a field. Note that for $\psi(x) = e(ax/p)$ and $\phi(x) = e(bx/p)$ for p a prime, the two notions agree.

While no explicit forms are known for such sums, sharp estimates are available. We have

Theorem 2.1 (A. Weil). *If ψ, ϕ are additive characters for a field of size $q = p^n$, $p \neq 2$, then*

$$|S(\psi, \phi)| \leq 2\sqrt{q}.$$

For general Kloosterman sums, we have

$$|S(a, b; c)| \leq \tau(c)(a, b, c)^{1/2} c^{1/2}$$

where $\tau(c)$ counts the number of divisors of c.

The proof of the above is involved, no matter what the case. Weil's proof requires some heavy machinery (in particular, class-field theory). Stepanov has an "elementary" proof, but the details are quite involved. We omit both, but note that one can find many generalizations of Kloosterman sums in the literature, and corresponding bounds resulting from the Riemann Hypothesis for finite fields.