either case, this implies that B contains at least one of its endpoints in violation of Lemma 2 and the proof is complete.

Clearly, the proofs given above are valid when A and B are arcs with or without one or both endpoints. We summarize these results in the following theorem.

THEOREM 1. *If A and B are nondegenerate connected subsets of arcs and if n is a positive integer, then there exist $(n, 1)$ functions f on A onto B when, and only when, A and B are open (i.e., both A and B lack both endpoints) and n is odd.*

The following theorem shows how limited the examples of $(n, 1)$ functions, n odd, must be. They are all about like the (3, 1) example given above.

THEOREM 2. *Let $n>0$ be an odd integer and let f be an $(n, 1)$ function on A onto B, where A and B are open intervals of the real line. For $b \in B, f^{-1}(b)$ contains as many relative maxima for f as it contains relative minima. In particular, f can not have an absolute maximum or absolute minimum on A.*

Proof. Suppose $f^{-1}(b)$ contains M relative maxima and m relative minima with either $m>0$ or $M>0$. If $m \neq M$ we may assume, without loss of generality, that $0 \leq m < M$. There are n numbers in $f^{-1}(b)$ but there is a number, b', slightly smaller than b such that $f^{-1}(b')$ contains at least $n-m+M$ numbers since there are M maxima and $f^{-1}(b')$ will have at least two elements near each of these maxima. However, $n-m+M>n$ since $M>m$ and this is a contradiction.

References

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ON A THEOREM OF USPENSKY

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Let α be a real number and define S_α to be the sequence $([\alpha], [2\alpha], [3\alpha], \cdots)$, where [ ] denotes the greatest integer function. The following result has been obtained by Uspensky [1]:

THEOREM. *Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are n positive real numbers which have the property that every positive integer occurs exactly once in some one of the sequences $S_{\alpha_1}, S_{\alpha_2}, \cdots, S_{\alpha_n}$. Then $n<3$.*

The proof given by Uspensky is somewhat elaborate and based on an approximation theorem of Kronecker. It is the purpose of this note to present a direct and elementary proof that $n<3$.

Proof. Assume that $n \geqq 3$ and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be n positive real numbers satisfying the hypothesis of the theorem. Certainly $\alpha_i > 1$ for all i and without loss of generality, we can assume that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Since 1 occurs in just one of the S_{α_i}, then it must occur in S_{α_1} and we have $[\alpha_1] = 1$. Thus $\alpha_1 = 1 + \delta$, where $0 < \delta < 1$.

Note that

$$\begin{aligned}[n\alpha_1] = l &\Rightarrow n\alpha_1 < l + 1 \\ &\Rightarrow (n+1)\alpha_1 = n\alpha_1 + \alpha_1 < l + 1 + 1 + \delta < l + 3 \\ &\Rightarrow [(n+1)\alpha_1] \leqq l + 2\end{aligned}$$

so that two consecutive integers cannot be missing from S_{α_1}.

Let m be the smallest positive integer which does not occur in S_{α_1}. Then m satisfies $(m-1)\delta < 1 \leqq m\delta$. By hypothesis, m must occur in some S_{α_i} and since $\alpha_2 < \alpha_3 < \cdots < \alpha_n$ then we have $[\alpha_2] = m$, i.e., $\alpha_2 = m + \epsilon$, where $0 \leqq \epsilon < 1$.

Now, let x be any integer which does not occur in S_{α_1}. Then x is of the form $[p\alpha_1] + 1$ for some p and $[(p+1)\alpha_1] - [p\alpha_1] > 1$. But

$$\begin{aligned}[(p+1)\alpha_1] - [p\alpha_1] > 1 &\Leftrightarrow [p + 1 + (p+1)\delta] - [p + p\delta] > 1 \\ &\Leftrightarrow [(p+1)\delta] - [p\delta] > 0 \\ &\Leftrightarrow p\delta < k\end{aligned}$$

and $(p+1)\delta \geqq k$ for some integer k.

Since $(p+m-1)\delta = p\delta + (m-1)\delta < k+1$ and $(p+m+1)\delta = (p+1)\delta + m\delta \geqq k+1$, there are two possibilities:

(1) $$(p+m)\delta \geqq k+1$$

in which case the *next* integer which does not occur in S_{α_1} is

$$[(p+m-1)\alpha_1] + 1 = p + m - 1 + k + 1 = p + k + m;$$

(2) $$(p+m)\delta < k+1$$

in which case the *next* integer which does not occur in S_{α_1} is $[(p+m)\alpha_1] + 1 = p + m + k + 1$.

Since $x = [p\alpha_1] + 1 = p + [p\delta] + 1 = p + k$, we have shown that if x is any integer which is missing from S_{α_1} then the next integer which is missing from S_{α_1} is either $x + m$ or $x + m + 1$.

Notice that

$$\begin{aligned}[n\alpha_2] = y &\Rightarrow n\alpha_2 - 1 < y \leqq n\alpha_2 \\ &\Rightarrow (n+1)\alpha_2 = n\alpha_2 + \alpha_2 < y + 1 + m + 1 = y + m + 2\end{aligned}$$

and $(n+1)\alpha_2 = n\alpha_2 + \alpha_2 \geqq y + m$. Therefore, $[(n+1)\alpha_2] = y + m$ or $y + m + 1$.

To complete the proof, suppose that the kth integer x_k which is missing from S_{α_1} is exactly the kth term $y_k = [k\alpha_2]$ of S_{α_2}. We have just shown that $x_{k+1} = x_k + m$ or $x_k + m + 1$ and $y_{k+1} = y_k + m$ or $y_k + m + 1$. But two consecutive integers cannot be missing from S_{α_1} and, by hypothesis, no integer can occur in both S_{α_1} and S_{α_2}. Consequently, we must have $x_{k+1}=y_{k+1}$. Since $x_1=m=y_1$, then by induction on k, we conclude that $x_n=y_n$ for all n. In other words, every positive integer occurs in either S_{α_1} or S_{α_2}. This is a contradiction to the assumption that $n \geqq 3$ and the proof is completed.

Reference

1. J. V. Uspensky, On a problem arising out of the theory of a certain game, this MONTHLY, 34 (1927) 516–521.

A REMARK ON SINGULAR STURM-LIOUVILLE DIFFERENTIAL EQUATIONS

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1. In 1934, Hahn [1] showed, as an *intermediate result*, that any system of orthogonal polynomials, whose first derivatives also form an orthogonal system, satisfies a differential equation of the form

$$p(x)y'' + s(x)y' + \lambda y = 0 \tag{1}$$

for which the coefficients are real and at most $p(x)$ is quadratic, $s(x)$ is linear, and λ is constant. As a *final result* he showed by examining the singularities that (1) is equivalent, up to a linear transformation, to the Hermite, Jacobi, or Laguerre differential equations.

We propose to arrive at Hahn's *final result* by a different method; the method and the results are given in section 2. We first state some definitions and known results that we shall need in our development.

We shall consider (1) as a singular Sturm-Liouville differential equation. It has been shown in [2] that (1) can be transformed into the self-adjoint form usually given as a Sturm-Liouville differential equation by multiplying through by the factor

$$T(x) = \frac{1}{p} \exp \int \frac{s}{p}\, dx. \tag{2}$$

It is known [3, 4] that, for a certain discrete set of values of λ, there exists over an interval (a, b) a set of solutions, called eigenfunctions, of the Sturm-Liouville problem consisting of (1) and a set of appropriate boundary conditions which are to be considered in the next section. Upon considering the orthogonality of these eigenfunctions over the interval, a weight function $w(x)$ is obtained. It follows from [2] that the weight function that corresponds to (1), and is customarily assumed to be positive over (a, b), is the same as (2). That is, $T(x)=w(x)$.

2. We start with the equation of Hahn's *intermediate result* (1) which may be written as

$$(\alpha x^2 + \beta x + \gamma)y'' + (\delta x + \epsilon)y' + \lambda y = 0. \tag{3}$$