THE SOLUTION OF A CERTAIN RECURRENCE

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In 1954, P. Turán [3] gave a proof of the identity

\[ \binom{n + p}{p} = \sum_{k=0}^{p} \binom{p}{k} \binom{n + 2p - k}{2p} \]

which he said appeared without proof in a book of the Chinese mathematician Le-Jen Shoo from 1867. This is equivalent to

\[ \binom{n}{p} = \sum_{k=0}^{p} \binom{p}{k} \binom{n + k}{2p} \]

or

\[ \binom{n}{m} = q_{nm} = \sum_{k=0}^{m} q_{mk} \binom{n + k}{2m}. \]  \hspace{1cm} (1)

In one of the many successors to Turán's paper T. S. Nandjundiah [2] noticed that the Shoo identity is an instance of the following expansion of a product of binomial coefficients, namely

\[ \binom{m}{p} \binom{n}{q} = \sum_{k=0}^{p} \binom{n - m + p}{p - k} \binom{m - n + q}{k} \binom{n + k}{p + q} \]  \hspace{1cm} (2)

(the upper limit of the sum is supplied by the convention that \(\binom{a}{b}\) is zero if \(a < 0, b < 0,\) or \(a < b\)). Let

\[ r_{nm} = \frac{1}{n + 1} \binom{n - m}{m} \binom{n + 1}{m + 1} = \frac{1}{m + 1} \binom{n}{m} \binom{n - 1}{m}. \]

These numbers appeared in a study of a telephone traffic system with inputs from two sources made by John P. Runyon and are known locally as Runyon numbers; cf. J. A. Morrison [1]. It follows from (2) that

\[ (m + 1)r_{nm} = \binom{n}{m} \binom{n - 1}{m} = \sum_{k=0}^{m+1} \binom{m + 1}{m-k} \binom{m - 1}{k} \binom{n + k}{2m} \]

or

\[ r_{nm} = \sum_{k=0}^{m+1} \frac{1}{m + 1} \binom{m + 1}{k + 1} \binom{m - 1}{k} \binom{n + k}{2m} = \sum_{k=0}^{m} r_{mk} \binom{n + k}{2m}, \]  \hspace{1cm} (3)

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a relation similar to (1). The natural question arising is: what is the general solution of

\[ x_{nm} = \sum_{k=0}^{m} x_{mk} \binom{n+k}{2m}. \]

Since the recurrence (4) leaves \( x_{nn} \) undetermined, this is the same as asking for the coefficient \( X_k(n, m) \) in

\[ x_{nm} = \sum_{k=0}^{m} X_k(n, m) x_{kk}. \]

The answer is given by the following

**Theorem.** If \( n = 0, 1, 2, \ldots, m = 0, 1, \ldots, n, \) and

\[ x_{nm} = \sum_{k=0}^{m} x_{mk} \binom{n+k}{2m}, \]

then

\[ x_{nm} = \sum_{k=0}^{m} \frac{2k + 1}{m + k + 1} \binom{n+k}{m+k+1} \binom{n-1-k}{m-k} x_{kk}, \quad \text{for} \ m < n \]

with arbitrary \( x_{kk}. \)

For a proof of the theorem, notice first that when \( x_{nm} = r_{nm}, x_{kk} = r_{kk} = \delta_{kk}, \) with \( \delta_{nm} \) the Kronecker delta; hence

\[ X_0(n, m) = r_{nm} = \frac{1}{m+1} \binom{n}{m} \binom{n-1}{m}. \]

Next, suppose that

\[ x_{nm} = \frac{2\rho + 1}{m + \rho + 1} \binom{n-1-\rho}{m-\rho} \binom{n+\rho}{m+\rho}, \quad \rho = 1, 2, \ldots, m. \]

Then, by (2)

\[ x_{nm} = \sum_{k=0}^{m} \frac{2\rho + 1}{m + \rho + 1} \binom{m-1-\rho}{k-\rho} \binom{m+\rho+1}{k+\rho+1} \binom{n+k}{2m} \]

\[ = \sum_{k=0}^{m} \frac{2\rho + 1}{k + \rho + 1} \binom{m-1-\rho}{k-\rho} \binom{m+\rho}{k+\rho} \binom{n+k}{2m} \]

\[ = \sum_{k=0}^{m} x_{mk} \binom{n+k}{2m} \]

while \( x_{kk} = \delta_{pk}; \) hence
\[ X_p(n, m) = \frac{2p + 1}{m + p + 1} \binom{n - 1 - p}{m - p} \binom{n + p}{m + p}, \quad p = 0, 1, \ldots, m \]

and the theorem is proved.

The theorem leads to binomial identities whenever a particular solution of (4) (for which \( x_{kk} \neq \delta_{pk}, \ p = 0, 1, \ldots, m \)) is known. Thus in the first instance \( x_{nm} = q_{nm} \) yields

\[
\binom{n}{m}^2 = \sum_{k=0}^{m} \frac{2k + 1}{m + k + 1} \binom{n - 1 - k}{m - k} \binom{n + k}{m + k} = \sum_{k=0}^{m} X_k(n, m)
\]

since \( q_{nn} = 1 \).

A direct proof of this identity is as follows. First

\[
\sum_{k=0}^{m} \frac{2k + 1}{m + k + 1} \binom{n - 1 - k}{m - k} \binom{n + k}{m + k}
\]

\[
= \sum_{k=0}^{m} \frac{2k + 1}{n - m} \binom{n - 1 - k}{m - k} \binom{n + k}{m + k + 1}
\]

\[
= \sum_{k=0}^{m} \frac{2m + 1}{n - m} \binom{n - 1 - k}{m - k} \binom{n + k}{m + k + 1}
\]

\[
- 2 \sum_{k=0}^{m} \frac{m - k}{n - m} \binom{n - 1 - k}{m - k} \binom{n + k}{m + k + 1}
\]

\[
= f_{nm} - g_{nm}.
\]

Next we have

\[
f_{nm} = \frac{2m + 1}{n - m} \sum_{k=0}^{m} \binom{n - m + k - 1}{k} \binom{n + m - k}{2m + 1 - k}
\]

\[
= \frac{2m + 1}{n - m} \sum_{k=0}^{m} \binom{n - m + k - 1}{k} \binom{n + m - k}{2m + 1 - k}
\]

\[
= \frac{2m + 1}{2n - 2m} \sum_{k=0}^{m} \binom{n - m + k - 1}{k} \binom{n + m - k}{2m + 1 - k}
\]

\[
= \frac{2m + 1}{2n - 2m} \binom{2n}{2m + 1} = \binom{2n}{2m}
\]

(the next to last step uses one form of the Vandermonde relation). Also

\[
g_{nm} = 2 \sum_{k=0}^{m} \binom{n - 1 - k}{m - 1 - k} \binom{n + k}{m + k + 1} = 2 \sum_{k=0}^{m} \binom{n - m + k}{k} \binom{n + m - 1 - k}{2m - k}
\]
and
\[
\binom{2n}{2m} = \sum_{k=0}^{2m} \binom{n-m+k}{k} \binom{n+m-1-k}{2m-k}
\]
\[
= \sum_{k=0}^{m-1} \binom{n-m+k}{k} \binom{n+m-1-k}{2m-k}
\]
\[
+ \sum_{k=0}^{m} \binom{n-m-1+k}{k} \binom{n+m-k}{2m-k}
\]
\[
= \frac{1}{2} \cdot q_{nm} = \sum_{k=0}^{m} \binom{n-m-1+k}{k} \left[ \binom{n+m-k-1}{2m-k} + \binom{n+m-k-2}{2m-k-1} + \cdots + \binom{n}{m+1} + \binom{n}{m} \right]
\]
\[
= q_{nm} + \binom{n}{m}^2
\]
which proves the identity.

Notice that
\[
(2m+1)^{-1}f_{nm} = (2m+1)^{-1} \binom{2n}{2m} = \sum_{k=0}^{m} \frac{1}{m+k+1} \binom{n-1-k}{m-k} \binom{n+k}{m+k}
\]
which is equation (5) with \(x_{kk} = (2k+1)^{-1}\); hence
\[
x_{nm} = (2m+1)^{-1} \binom{2n}{2m}
\]
is a solution of (4) and
\[
\frac{1}{2m+1} \binom{2n}{2m} = \sum_{k=0}^{m} \frac{1}{2k+1} \binom{2m}{2k+1} \binom{n+k}{2m}
\]
or
\[
\binom{2n}{2m} = \sum_{k=0}^{m} \binom{2m+1}{2k+1} \binom{n+k}{2m}
\]
which is the \(x_{nm}\) with \(x_{kk} = 2k+1\). Since sums and differences of solutions of (4)
are also solutions, it follows that

$$x_{nm} = \frac{1}{2} \left( \binom{2n+1}{2m} - \binom{n}{m} \right)$$

is the solution for which $x_{kk} = k$.

References

2. T. S. Nandjundiah, Remark on a note of P. Turán, this MONTHLY, 65 (1958) 354.

ON THE TOTIENT FUNCTIONS OF JORDAN AND ZSIGMONDY

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Introduction. K. Zsigmondy (see [2], p. 152) devised a function to determine the number of elements of a certain order in a finite abelian group.

In this note it will be shown that Zsigmondy's function can be described completely by use of Jordan's totient function (see [2], p. 147). The proof is elementary and is much simpler than the lengthy combinatorial proofs of the formula found in the literature (see, for example, [1]).

I. In order to translate the problem into number-theoretic concepts, we make the following definitions:

Definition. Let $n$ and $k$ be positive integers. A $k$-tuple $\{a_1, a_2, \ldots, a_k\}$ of positive integers is called a prime sequence for $n$ (of length $k$) provided $1 \leq a_i \leq n$ and $(a_1, a_2, \ldots, a_k, n) = 1$ (the parentheses denote the greatest common divisor).

Definition. If $n$ and $k$ are positive integers, then $J_k(n)$ denotes the number of distinct prime sequences for $n$, each of length $k$. $J_k(n)$ is defined to be zero.

Definition. Let $m$, $n_1$, $n_2$, \ldots, $n_s$ be fixed positive integers. An $s$-tuple $\{a_1, a_2, \ldots, a_s\}$ of positive integers is called a primitive sequence for $m$ (with respect to $n_1, \ldots, n_s$) provided

(1) $1 \leq a_i \leq n_i$ \hspace{1em} (i = 1, 2, \ldots, s) and

(2) $m$ is the smallest positive integer such that $ma_i \equiv 0 \pmod{n_i}$ \hspace{1em} (i = 1, 2, \ldots, s).

Definition. If $m$ is a positive integer then $\psi(m) = \psi(m; n_1, n_2, \ldots, n_s)$ denotes the number of distinct primitive sequences for $m$ (with respect to $n_1, n_2, \ldots, n_s$).

Thus if $G$ is a finite abelian group with independent generators $g_1, g_2, \ldots, g_r$ of order $n_1, n_2, \ldots, n_r$, respectively, then $\psi(m)$ is the number of elements of $G$ of order $m$.

II. Theorem. $\psi$ is a multiplicative function.