RAMSEY'S THEOREM FOR \( n \)-DIMENSIONAL ARRAYS

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Introduction. An analogue to a theorem of Ramsey [5] has been conjectured for finite vector spaces by Gian-Carlo Rota. Namely, for each choice of positive integers \( k, l, r \), and finite field \( F = GF(q) \), there exists an integer \( N(k, l, r; q) \) such that if \( n \geq N(k, l, r; q) \) and the \( k \)-dimensional subspaces of an \( n \)-dimensional vector space \( V \) over \( F \) are partitioned into \( r \) classes, then some \( l \)-dimensional subspace of \( V \) has all of its \( k \)-dimensional subspaces in one class. In this note we present a very general theorem of this type, a brief outline of its proof, and general applications, including some cases of Rota's Conjecture. Complete details will appear elsewhere.

Notation. Let \( A = \{a_1, \ldots, a_t\} \) be a finite set with \( t > 1 \) and let \( H_p: A \rightarrow A \) be a permutation group on \( A \). Define \( H_c = \{a \in A\} \) to be the set of maps of \( A \) into \( A \) given by \( x^a = a \) for all \( x \in A \). \( H \) will denote \( H_c \cup H_p \). We can define an action of \( H \) on \( A^t \) by \((x_1, \ldots, x_t)^\sigma = (x_1^{\sigma}, \ldots, x_t^{\sigma})\) for \( x_i \in A, \sigma \in H \). Let \( I_0 \) denote \( \{a_1, \ldots, a_t\} \in A^t \) and let \( L_c = \{l_0: \sigma \in H_c\}, L_p = \{l_0: \sigma \in H_p\}, L = L_c \cup L_p \). We introduce the basic concept of a \( k \)-parameter set. For fixed nonnegative integers \( k \leq n \), let \( \Pi = \{S_0, S_1, \ldots, S_k\} \) be a partition of the set \( I_n = \{1, 2, \ldots, n\} \) with \( S_i \neq \emptyset \) for \( 1 \leq i \leq k \). \( S_0 = \emptyset \) is possible. Let \( f: I_n \rightarrow H \) have the property

\[
 f(i) \in H_c \text{ if } i \in S_0, \\
 f(i) \in H_p \text{ otherwise.}
\]

The set \( P(\Pi, f) \) is defined by

\[
P(\Pi, f) = \bigcup_{1 \leq i_0 < i_1 < \cdots < i_t \leq n} \{(x_1, \ldots, x_t); x_j = a_{f(j)}^{i_j} \text{ if } j \in S_{i_j}\} \subseteq A^n.
\]

Note that since \( f(j) \in H_c \) for \( j \in S_0 \), \( P(\Pi, f) \) consists of exactly \( t^k \) elements of \( A^n \).

Definition. \( P_k \) is a \( k \)-parameter set of \( A^n \) if and only if \( P_k = P(\Pi, f) \) for some partition \( \Pi \) and mapping \( f \). Of course, we say that \( P_k \) is a \( k \)-parameter subset of the \( l \)-parameter set \( P_l \subseteq A^n \) if \( P_k \subseteq P_l \) and \( P_k \) is a \( k \)-parameter set of \( A^n \).

The main results.

Theorem 1. For each choice of positive integers \( k, l, r \) there exists an
integer $M(k, l, r)$ such that if $m \geq M(k, l, r)$ and the $k$-parameter subsets of an $m$-parameter set $P_m \subseteq \mathcal{A}^n$ are partitioned into $r$ classes, then there exists an $l$-parameter subset $P_1 \subseteq P_m$ such that all $k$-parameter subsets of $P_1$ belong to the same class.

Let us call a $k$-parameter set $P_k \subseteq \mathcal{A}^n$ normalized if $f(j) = \sigma_1$ for all $j \in S_0$. We state the important

**Theorem 2.** The preceding theorem is valid if all parameter sets are required to be normalized.

Before proceeding to the proof outline, we list several immediate corollaries to the theorems.

**Corollary 1.** Given integers $k$ and $r$, there exists an integer $N(k, r)$ such that if $|A| \geq N(k, r)$ and the finite subsets of $A$ are partitioned into $r$ classes then there exist $k$ disjoint nonempty subsets $A_1, \ldots, A_k$ of $A$ such that all $2^k - 1$ unions $\bigcup_{j \in J} A_j$, $\emptyset \neq J \subseteq \{1, 2, \ldots, k\}$, is $I_k$, are in the same class.

This follows from Theorem 2, taking $A = \{0, 1\}$ and $H_r = \{e\}$.

**Corollary 2 (J. Folkman, J. Sanders [6]).** Given integers $k$ and $r$, there exists an integer $N(k, r)$ such that if $n \geq N(k, r)$ and the set $I_n$ is partitioned into $r$ classes, then there exist $k$ integers $a_1, \ldots, a_k$ such that all sums $\{\sum_{i=1}^k \epsilon_i a_i : \epsilon_i = 0 \text{ or } 1, \text{ not all } \epsilon_i = 0\}$ are in the same class.

This follows for Corollary 1 by interpreting the characteristic function of $A_i$ as the dyadic expansion of an integer $a_i$. For $k = 2$, Corollary 2 was first proved by Schur [7]. Schur's result can also be stated as follows:

Given $r$, there exists an integer $N(r)$ such that if $n \geq N(r)$ and the set $I_n$ is partitioned into two classes, then the equation $x + y = z$ can be solved in one class. This is also a special case of

**Corollary 3.** Let $\mathcal{E} = L_i(x_1, \ldots, x_m)$, $1 \leq i \leq n$ be a system of homogeneous linear equations with the property that for each $j$, $1 \leq j \leq m$, there exists a solution $(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$ to the system $\mathcal{E}$ with $\epsilon_i = 0$ or $1$ and $\epsilon_j = 1$. Then given an integer $r$ there exists an integer $N(r)$ such that if $n \geq N(r)$ and the set $I_n$ is partitioned into $r$ classes, then $\mathcal{E}$ can be solved in one class.

This is similar to a result of R. Rado [3].

**Corollary 4 (van der Waerden [2]).** Given integers $k$ and $r$ there exists an integer $N(k, r)$ such that if $n \geq N(k, r)$ and the set $I_n$ is partitioned into $r$ classes, then at least one class contains an arithmetic progression of length $k$. 

This result is implied by the stronger

**Corollary 5 (Hales-Jewett [1])**. Let \( A = \{a_1, \ldots, a_t\} \) be a finite set. Given an integer \( r \) there exists an integer \( N(r, l) \) such that if \( n \geq N(r, l) \) and the set \( A^n \) is partitioned into \( r \) classes, then there exists a set of \( t \) elements of the form

\[
X_i = (x_{1i}, \ldots, x_{1u}, a_i, x_{2i}, \ldots, x_{2v}, a_i, \ldots, a_i, x_{di}, \ldots, x_{du}) \in A^n,
\]

\( 1 \leq i \leq t, \)

all of which belong to one class.

This follows from Theorem 1 by taking \( A = \{a_1, \ldots, a_t\}, k = 0, l = 1, H_p = \{e\} \).

**Corollary 6.** Given integers \( l \) and \( r \) and a finite field \( GF(q) \) there exists an integer \( N(l, r, q) \) such that if \( n \geq N(l, r, q) \) and the 1-dimensional subspaces of an \( n \)-dimensional vector space \( V \) over \( GF(q) \) are partitioned into \( r \) classes, then \( V \) contains an \( l \)-dimensional subspace \( V' \) all of whose 1-dimensional subspaces are in one class.

This follows from Theorem 2 by taking \( A = GF(q), H_p = \text{mult. group of } GF(q), k = 0 \). The corresponding result for affine spaces over \( GF(q) \) follows from Theorem 1. Corollary 6 was first proved for \( q = 2 \) by D. Kleitman (unpublished) and \( q = 3, 4 \) by B. L. Rothschild [4]. From the result for 1-dimensional affine subspaces, techniques of Rothschild [4] can be used to prove the result corresponding to Corollary 6 when 1-dimensional subspace is replaced by 2-dimensional subspace. It was conjectured by G.-C. Rota that Corollary 6 holds for \( k \)-dimensional subspaces in general.

Finally, as a more powerful application, let \( C^n \) denote an \( n \)-dimensional cube in \( E^n \). Let us say that a set \( S_k \) of \( 2^k \) vertices of \( C^n \) forms a \( k \)-subspace of \( C^n \) if \( S_k \) is contained in some \( k \)-dimensional Euclidean subspace of \( E^n \).

**Corollary 7.** Given integers \( k, l, r \) there exists an integer \( N(k, l, r) \) such that if \( n \geq N(k, l, r) \) and the \( k \)-subspaces of \( C^n \) are partitioned into \( r \) classes, then there exists an \( l \)-subspace of \( C^n \) all of whose \( k \)-subspaces are in one class.

**Brief outline of proof of Theorem 1.** Let \( S(k; t_1, \ldots, t_r) \) denote the statement:

There exists an integer \( M(k; t_1, \ldots, t_r) \) such that if \( m \geq M(k; t_1, \ldots, t_r) \) and the \( k \)-parameter subsets of an \( m \)-parameter set \( P_m \) are partitioned into \( r \) classes \( C_1, C_2, \ldots, C_r \), then there exists
an $t$, $1 \leq t \leq r$ and an $t$-parameter subset $P_u$ of $P_m$ such that all the $k$-parameter subsets of $P_u$ belong to class $C_l$.

We prove $S(k; t_1, \ldots, t_r)$ by multiple induction on $k$ and $t_1 + t_2 + \cdots + t_r$. We can assume $0 \leq k$, $r \geq 1$ and $t_i \geq 1$ for all $i$. The first step in the induction is $S(0; t_1, \ldots, t_r)$. Once certain notational difficulties have been overcome, the proof of this statement is relatively straightforward. We assume $S(i; t_1, \ldots, t_r)$ has been established for $0 \leq i < k$ and all $t_i$. Since $S(k; t_1, \ldots, t_r)$ is certainly valid if $t_1 + t_2 + \cdots + t_r \leq rk$, we further assume that for some $t > rk$, $S(k; t_1, \ldots, t_r)$ is valid for all choices of $t_i$ with $t_1 + \cdots + t_r < t$.

A critical step in the proof rests on the following fact. It is possible to define a map $M: L^n \rightarrow 2^a$ such that for each $l$-parameter set $P_i \subseteq A^n$ there exists an $(l-1)$-parameter set $P_{i-1} \subseteq L^n$ with $M(P_{i-1}) = P_i$ such that for "certain" $k$-parameter subsets $P_k \subseteq P_i$, there exists a $(k-1)$-parameter subset $P_{k-1} \subseteq P_{i-1}$ which makes the following diagram commutative:

$$
P_{k-1} \subseteq P_{i-1} \subseteq \quad M \downarrow \quad \downarrow M
P_k \quad \subseteq P_i
$$

Thus, the original partition of the $k$-parameter sets $P_k$ into $r$ classes induces a partition of $(k-1)$-parameter sets $P_{k-1}$ to which we can apply the induction hypothesis. It turns out that the "remaining" $k$-parameter sets can be naturally embedded in a large parameter set to which we can again apply the preceding argument. After a large number of iterations of this procedure, we are left with a configuration of blocks of "remaining" $k$-parameter sets which in a certain sense is isomorphic to a large parameter set in which the blocks are identified with points. By then partitioning these point-blocks according to the way in which the corresponding constituent $k$-parameter subsets have been partitioned and applying $S(0; t_1', \ldots, t_r')$ for suitable $t_1', \ldots, t_r'$, we can extract a configuration of $k$-parameter sets from which the induction step can be completed fairly directly. Theorem 2 follows from Theorem 1 with little difficulty. As might be expected, the bounds provided on $M(k, l, r)$ by this proof are extremely large.

References

7. I. Schur, *Über die Kongruenz $x^m + y^m = z^m$ (mod $p$)*, Jahr. Deutsch. Math.-Verein. 25 (1916), 114.

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