A PACKING INEQUALITY FOR COMPACT CONVEX
SUBSETS OF THE PLANE

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1. Introduction. Let $X$ be a compact metric space. By a packing in $X$ we mean a subset $S \subseteq X$ such that, for $x, y \in S$ with $x \neq y$, the distance $d(x, y) \geq 1$. Since $X$ is compact, any packing of $X$ is finite. In fact, the set of numbers

$$\{ \text{card}(S) : S \text{ is a packing in } X \}$$

is bounded. The cardinality of the largest packing in $X$ will be called the packing number of $X$ and will be denoted by $\rho(X)$. If $A(X)$ and $P(X)$ denote the area and perimeter, respectively, of a compact convex subset $X$ of the plane, then a special case of a result conjectured by H. Zassenhaus [6] and proved by N. Oler [1] is the following.

**THEOREM (Oler).**

$$\rho(X) \leq \frac{2}{\sqrt{3}} A(X) + \frac{1}{2} P(X) + 1. \quad (1)$$

Unfortunately, Oler's proof of his general theorem requires 30 pages of rather detailed arguments. It is our purpose in this note to establish a theorem of this type for simplicial complexes in the plane. This theorem will imply (1) and, moreover, the arguments used are quite elementary.

2. Preliminaries. By a $p$-simplex in the plane we mean the convex hull of $p + 1$ points in general position in the plane. Since there can be at most 3 points in general position in the plane, we must have $p = 0, 1$ or 2. If $x_0, \ldots, x_p$ are in general position, $(x_0, \ldots, x_p)$ will denote the $p$-simplex which is their convex hull. The points $x_0, \ldots, x_p$ will be called the vertices of $(x_0, \ldots, x_p)$.

If $\sigma$ and $\tau$ are simplexes, we say that $\sigma$ is a face of $\tau$ if the vertices of $\sigma$ are a subset of the vertices of $\tau$.


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By a simplicial complex in the plane, we mean a finite set $K$ of the simplexes in the plane with the following properties:

(2) if $\sigma \in K$ then every face of $\sigma$ is in $K$;

(2') if $\sigma, \tau \in K$ and $\sigma \cap \tau$ is nonempty, then $\sigma \cap \tau$ is a face of both $\sigma$ and $\tau$.

Let $K$ be a simplicial complex in the plane. We denote by $\vert K \vert$ the union of the simplexes in $K$. If $r \geq 0$ is an integer, we denote by $K^r$ the set of all $p$-simplexes in $K$ with $p \leq r$. We let $\alpha_r(K)$ denote the number of $r$-simplexes in $K$. The Euler characteristic $\chi(K)$ is defined by $\chi(K) = \alpha_0(K) - \alpha_1(K) + \alpha_2(K)$.

It is a theorem of combinatorial topology that $\chi(K)$ depends only on $\vert K \vert$ (cf. [5]).

If $\sigma$ is a 1-simplex in $K$ we let $\varepsilon(\sigma, K)$ be the number of 2-simplexes in $K$ having $\sigma$ as a face. By (2'), $\varepsilon(\sigma, K) \leq 2$. If $\sigma$ is a 1-simplex or a 2-simplex in the plane, we let $m(\sigma)$ denote the length or the area, respectively, of $\sigma$. We define $A(K)$ and $P(K)$ by

$$A(K) = \sum_{\sigma \in K^2} m(\sigma)$$

$$P(K) = \sum_{\sigma \in K^2} \left( 2 - \varepsilon(\sigma, K) \right) m(\sigma).$$

The numbers $A(K)$ and $P(K)$ depend only on $\vert K \vert$ since $A(K)$ is the area of $\vert K \vert$ while $P(K)$ is its perimeter (suitably defined).

3. The Main Result.

THEOREM. Let $K$ be a simplicial complex in the plane. Suppose that for any $x, y \in K^0$ with $x \neq y$ we have $d(x, y) \geq 1$. Then

$$\alpha_0(K) \leq \frac{2}{-\sqrt{3}} A(K) + \frac{1}{2} P(K) + \chi(K).$$

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The proof of the theorem will be by induction using the following two lemmas.

**LEMMA 1.** Let \( \triangle \) be a triangle with area \( A \) and sides of length \( s_1, s_2, \text{ and } s_3 \). If \( s_1 > s_2 > s_3 > 1 \) then \( \frac{4}{\sqrt{3}} A + s_1 > s_2 + s_3 \).

Proof. We first note

\[
\frac{(s_1 + s_2 + s_3)(s_1 + s_2 - s_3)(s_1 - s_2 + s_3)}{4} \geq \frac{s_1 + s_2 + s_3}{3} \geq \frac{3}{3} = \frac{1}{3}(s_2 + s_3 - s_1). 
\]

Using Hero's formula for the area of a triangle together with the inequalities \( s_1 \leq s_2 + s_3 \) and \( A \geq 0 \) we obtain

\( 16A^2 \geq 3(s_2 + s_3 - s_1)^2 \). Hence \( 4A \geq \frac{3}{3} (s_2 + s_3 - s_1) \), or \( \frac{4}{\sqrt{3}} A + s_1 > s_2 + s_3 \) as required.

**LEMMA 2.** Let \( Q \) be a convex quadrilateral in the plane with area \( A \) and perimeter \( P \). Suppose that:

- length of any diagonal of \( Q \) > length of any side of \( Q \) > 1.

Then \( \frac{4}{\sqrt{3}} A - P + 2 > 0 \).

Proof. The sum of the interior angles of \( Q \) is \( 2\pi \), so one pair of diagonally opposite angles must have sum \( \leq \pi \). We assume that this is the pair labelled \( \theta \) and \( \theta' \) in Fig. 1.

![Figure 1](image)

Now \( d \geq s, t \) so \( \theta \) is the largest angle in the triangle with sides labelled \( s, t \) and \( d \). Therefore, \( \theta \geq \pi/3 \). Similarly \( \theta' \geq \pi/3 \). But \( \theta + \theta' \leq \pi \), so \( \pi/3 \leq \theta, \theta' \leq 2\pi/3 \). It follows that
\[ A = \frac{1}{2} (st \sin \theta + s't' \sin \theta') \geq \frac{\sqrt{3}}{4} (st + s't'). \]

Hence,

\[ \frac{4}{\sqrt{3}} A - P + 2 \geq st + s't' - (s + t + s' + t') + 2 = (s-1)(t-1) + (s'-1)(t'-1) \geq 0 \]

since \( s,t,s',t' \geq 1. \)

**Proof of Theorem.** Suppose \( K \) contains only one simplex. Then that simplex is a \( 0 \)-simplex and \( A(K) = P(K) = 0. \) The inequality (3) reduces to \( \sigma_0(K) = 1 = \chi(K). \)

Now suppose that \( K \) contains more than one simplex and that the theorem holds for all complexes with fewer simplexes than \( K. \) Let \( \mathcal{U} \) be the class of all complexes \( L \) in the plane such that \( |L| = |K| \) and \( L^0 = K. \) Every member of \( \mathcal{U} \) satisfies the hypothesis of the theorem. Furthermore, since the numbers occurring in (3) depend only on \( |K| \) and \( K^0, \) to establish (3) for \( K \) it suffices to establish it for any member of \( \mathcal{U}. \) Henceforth we shall assume that \( K \) is chosen from \( \mathcal{U} \) so that

\[ \sum_{\sigma \in K \setminus K^0} m(\sigma) \]

is minimal.

Suppose \( K \) contains no \( 1 \)-simplexes. Then \( K \) contains only \( 0 \)-simplexes, \( A(K) = P(K) = 0, \) and (3) reduces to \( \sigma_0(K) = \chi(K). \)

Finally, suppose \( K \) contains a \( 1 \)-simplex. Let \( \sigma \) be a \( 1 \)-simplex in \( K \) with \( m(\sigma) \) as large as possible.

**Case I.** \( \epsilon(\sigma, K) = 0. \) Then \( K - \{ \sigma \} \) is a complex. By the inductive assumption,
\[ \alpha_0(K) = \alpha_0(K - \{ \sigma \}) \leq \frac{2}{\sqrt{3}} A(K - \{ \sigma \}) + \frac{1}{2} P(K - \{ \sigma \}) + \chi(K - \{ \sigma \}) \]
\[ = \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) - m(\sigma) + \chi(K) + 1 \]
\[ \leq \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) + \chi(K). \]

**Case II.** \( \varepsilon(\sigma, K) = 1. \) Let \( \tau \) be the 2-simplex in \( K \) having \( \sigma \) as a face. Let \( \sigma' \) and \( \sigma'' \) be the other one-dimensional faces of \( \tau \), where we can assume \( m(\sigma') \geq m(\sigma'') \) without loss of generality. By the hypothesis of the theorem and the choice of \( \sigma \), we have

\[ m(\sigma) \geq m(\sigma') \geq m(\sigma'') \geq 1. \]

By Lemma 1,

\[ \frac{4}{\sqrt{3}} m(\tau) + m(\sigma) \geq m(\sigma') + m(\sigma''). \]

Since \( \tau \) is the only 2-simplex having \( \sigma \) as a face, the collection \( L = K - \{ \sigma, \tau \} \) is a complex. By the inductive assumption and (4) we have

\[ \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) + \chi(K) \]
\[ = \frac{2}{\sqrt{3}} A(L) + \frac{1}{2} P(L) + \chi(L) + \frac{2}{\sqrt{3}} m(\tau) + \frac{1}{2} (m(\sigma) - m(\sigma') - m(\sigma'')) \]
\[ \geq \frac{2}{\sqrt{3}} A(L) + \frac{1}{2} P(L) + \chi(L) \geq \alpha_0(L) = \alpha_0(K). \]

**Case III.** \( \varepsilon(\sigma, K) = 2. \) Let \( \tau_1 \) and \( \tau_2 \) be the 2-simplexes in \( K \) with \( \sigma \) as a face. We shall first show that \( Q = \tau_1 \cup \tau_2 \) is a convex quadrilateral satisfying the hypotheses of Lemma 2.
Let $X$ and $Y$ be the vertices of $\sigma$ and let $X, Y, Z$ and $X, Y, W$ be the vertices of $\tau_1$ and $\tau_2$ respectively. The sides of $Q$ are the 1-simplexes $(X, Z), (X, W), (Y, Z), (Y, W)$ which by the hypothesis of the theorem and the choice of $\sigma$ all have length $\geq m(\sigma)$. Hence, $Z$ and $W$ must lie in the region $R$ shown in Fig. 2. $R$ is bounded by two circular arcs of radius $m(\sigma)$ with centers at $X$ and $Y$, and $R$ is bisected by $\sigma$. By $(2')$, $Z$ and $W$ must lie on opposite sides of $\sigma$; hence, $Q$ is convex.

![Figure 2](image)

In order to show the hypotheses of Lemma 2 are satisfied, it remains to show that the diagonal of $Q$ from $Z$ to $W$ is at least as long as any side of $Q$. Suppose the contrary. Then $m(Z, W) < m(\sigma)$. Let $\sigma' = (Z, W), \tau'_1 = (X, Z, W)$ and $\tau'_2 = (Y, Z, W)$.

The collection

$$L = (K - \{\sigma, \tau'_1, \tau'_2\}) \cup \{\sigma', \tau'_1, \tau'_2\}$$

is a complex in $K$. But

$$\sum_{\lambda \in L - L}^{\lambda \in K - K} 0 m(\lambda) = \sum_{\lambda \in L - L}^{\lambda \in K - K} 0 m(\lambda) - m(\sigma) + m(\sigma') < \sum_{\lambda \in K - K}^{\lambda \in K - K} m(\lambda)$$

contradicting the choice of $K$ from the class $\mathcal{H}$. We now apply Lemma 2 to obtain

$$(5) \quad \frac{4}{\sqrt{3}} (m(\tau'_1) + m(\tau'_2)) = (m(X, Z) + m(X, W) + m(Y, Z) + m(Y, W)) + 2 \geq 0.$$
Let $M = K - \{\sigma, \tau_1, \tau_2\}$. Then $M$ is a complex with fewer simplexes than $K$. By the inductive assumption and (5) we have

$$\frac{2}{\sqrt{3}}A(K) + \frac{1}{2}P(K) + \chi(K)$$

$$= \frac{2}{\sqrt{3}}A(M) + \frac{1}{2}P(M) + \chi(M) + \frac{2}{\sqrt{3}}(m(\tau_1) + m(\tau_2))$$

$$- \frac{1}{2}(m(X, Z) + m(X, W) + m(Y, Z) + m(Y, W)) + 1$$

$$\geq \frac{2}{\sqrt{3}}A(M) + \frac{1}{2}P(M) + \chi(M) \geq \sigma_0(M) = \sigma_0(K).$$

This completes the proof of the theorem.

To show that (3) implies (1) we argue as follows. Let $X$ be a convex compact subset of the plane. Let $S$ be a packing of $X$ with $\text{card}(S) = \rho(X)$. Let $Y$ denote the convex hull of $X$ so $A(Y) \leq A(X)$ and $P(Y) \leq P(X)$. Let $K$ be a complex with $K^0 = S$ and $|K| = Y$. (The existence of such a complex is easily seen, for example, by induction on the number of points in $S$.)

Since $S$ is a packing, $K$ satisfies the hypotheses of the theorem. Since $|K| = Y$ is convex, $\chi(K) = 1$. By (3),

$$\rho(X) = \text{card}(S) = \sigma_0(K)$$

$$\leq \frac{2}{\sqrt{3}}A(K) + \frac{1}{2}P(K) + \chi(K)$$

$$\leq \frac{2}{\sqrt{3}}A(X) + \frac{1}{2}P(X) + 1$$

which is (4).

4. Concluding remarks. It was pointed out by Oler [2] that (4) can be used to establish the following result suggested by P. Erdős: If $T_n$ denotes the regular 2-simplex of side $n$ then

$$\rho(T_n) = \binom{n+2}{2}.$$  

The $m$-dimensional analogues ($m \geq 3$) of (3) have not yet been

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found. Indeed, if \( T_n^{(m)} \) denotes the regular \( m \)-simplex of edge length \( n \), it is not known that

\[
\rho \left( T_n^{(m)} \right) = \binom{m+n}{m}.
\]

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