ON SMALL GRAPHS WITH FORCED MONOCHROMATIC TRIANGLES

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Let us denote by $S(k,t; r)$ the following statement:

There exists a graph $G$ which does not contain a complete subgraph on $t$ vertices but which has the property that any $r$-coloring of the edges of $G$ must contain a monochromatic complete subgraph on $k$ vertices.

It is immediate from Ramsey's Theorem (cf. [5]) that for any fixed $k$ and $r$, $S(k,t; r)$ is true for $t$ sufficiently large. In particular, it follows that $S(3,7; 2)$ holds by taking $G$ to be $K_6$, the complete graph on 6 vertices. Recently, Erdős and Hajnal [1] asked whether $S(3,6; 2)$ holds. This was first answered affirmatively by J. H. van Lint (unpublished) who gave as an example of a graph which establishes $S(3,6; 2)$, the complement of the graph shown in Fig. 1.

![Graph Diagram](image)

Figure 1

Soon thereafter, L. Pósa (unpublished) proved the existence of a graph $G$ for which $S(3,5; 2)$ holds, basing his work on some previous existence proofs of Erdős.
The final step in this direction was achieved by the late J. H. Folkman [2] who established \( S(3,4; 2) \) by the explicit construction of an appropriate (very large) graph \( G \). More generally, Folkman also established \( S(k,k+1; 2) \) in [2] for all \( k \geq 3 \). Furthermore, Folkman asserted in 1968 that he had a proof of \( S(3,4; 3) \) and a very complicated proof of \( S(3,4; 4) \) but no notes on these ideas have as of yet been discovered. It was conjectured by Folkman and independently by Erdős and Hajnal that \( S(k,k+1; r) \) holds for all \( k \) and \( r \).

Erdős has pointed out that it would be of interest to determine the least number \( N(k,t; r) \) of vertices a graph may have which can be used to establish \( S(k,t; r) \). It was shown by one of the authors in [3] that \( N(3,6; 2) = 8 \). The unique graph \( G \) which achieves this bound is the complement of the 8 vertex graph shown in Fig. 2. Thus, \( G \) has 8 vertices and 23 edges.

![Figure 2](image1)

The results of [2] show that \( N(3,4; 2) < \infty \). In a recent paper, Schäuble [6] proves \( N(3,5; 2) \leq 42 \) by considering the graph shown in Fig. 3.

![Figure 3](image2)
Here, we use the notation

\[ G \rightarrow \rightarrow H \]

to indicate that all vertices of G are connected to all vertices of H.

In this note we prove the following result:

**Theorem:** \( N(3,5; 2) \leq 23 \).

**Proof:** Consider the graph G given in Fig. 4.

![Figure 4](image_url)

In G, each vertex of pentagon A is just connected to the vertices \( t_2 \) and \( t_3 \) of triangle T, each vertex of B is connected to vertices \( t_1 \) and \( t_2 \) of T, and each vertex of C is connected to vertices \( t_1 \) and \( t_3 \). All vertices of pentagon X are connected to all vertices of pentagons A, B, C. Thus, G has 23 vertices and 128 edges. We must show that G can be used to establish \( S(3,5; 2) \).

1. \( K_5 \subseteq G \). Consider the possible locations of the vertices of a hypothetical subgraph \( K_5 \). We cannot have \( \geq 3 \) vertices of this \( K_5 \) in one pentagon A, B, C or X since they all contain no triangles. Also, since there are no edges between pentagons A, B and C, no vertex of the \( K_5 \) can be in X. If the \( K_5 \) had \( \geq 3 \) vertices not in T, at least two of the pentagons A, B, C would have to contain a vertex of the \( K_5 \)
which is impossible since these pentagons have no interconnecting edges. The only possibility left is if all 3 vertices of $T$ were also vertices of the $K_5$. The remaining 2 vertices of the $K_5$ must then belong to one of $A$, $B$, $C$ which is also impossible.

(ii) Any 2-coloring of the edges of $G$ contains a monochromatic triangle. We need two preliminary facts to establish (ii). We refer to Fig. 5 for the graphs under consideration. Assume the graphs $H_1$ and $H_2$ have been 2-colored so that no monochromatic triangles have been formed.

![Figure 5](image)

(a) All edges of the pentagons $P$ and $Q$ of $H_1$ must be the same color. This fact was used by Schäuble in [6]. We indicate a short proof. Assume some edge $e$ of $P$ is red. If $\geq 3$ of the edges from some endpoint $p_1$ of $e$ to $Q$ were red then 2 of these edges must go to adjacent vertices of $Q$, say, $q_1$ and $q_2$. But if any edge between $p_2, q_1, q_2$ is red then we get a red triangle; if they are all blue then we get a blue triangle. Thus, at most 2 of the edges from $p_1$ to $Q$ can be red, i.e., at least 3 of them are blue. Of course, this is also true for the other endpoint of $e$. But this implies that any edge of $P$ adjacent to $e$ must also be red since they share a common endpoint. Hence, all edges of $P$ are red. Hence, at least $3/5$ of all the edges between $P$ and $Q$ must be blue which implies by symmetry that all the edges of $Q$ are also red. This proves (a).

(b) If all edges of pentagon $R$ of $H_2$ are red then the edge $f$ is red. Assume $f$ is blue. For each vertex $r$ of $R$ consider the ordered pair of colors $(C_x(r), C_y(r))$ where $C_x(r)$ is the color assigned to the edge from $r$ to $x$, with $C_y(r)$ defined similarly. We certainly cannot have $(C_x(r), C_y(r)) = (\text{blue}, \text{blue})$ since this forms a blue triangle $r, x, y$. Also $(C_x(r), C_y(r)) = (\text{red}, \text{red})$ is impossible because any red edge between $r', x, y$ forms a red triangle and if these edges are all blue then a blue triangle is formed. Hence, we must have $(C_x(r), C_y(r)) = (\text{red}, \text{blue})$ or $(\text{blue}, \text{red})$. However, we cannot have $(C_x(r), C_y(r)) = (C_x(r'), C_y(r'))$ because the red component, say, $C_x(r) = C_x(r') = \text{red},$
would form a red triangle $r,r',x$. Hence adjacent vertices in $H_2$ must have distinct pairs $(C_x(r), C_y(r))$. This is impossible however because $H_2$ is an odd cycle. This proves (b).

The proof of (11) is now immediate. Assume without loss of generality that some edge of pentagon $X$ in $G$ is red. Hence by (a), all edges of $A$, $B$ and $C$ are also red. Finally, by (b), all edges of triangle $T$ are red. This proves the Theorem.

It might be conjectured that $N(3,5; 2) = 23$ although admittedly there is not too much evidence for such an assertion. It seems very difficult to establish any nontrivial lower bounds on the $N(k,l; r)$. S. Lin [4] has recently shown $N(3,5; 2) \geq 10$. However, it is not known even if $N(3,5; 2) \geq 11$.

REFERENCES