ON SUMS OF FIBONACCI NUMBERS

P. ERDÖS
Hungarian Academy of Sciences, Budapest, Hungary and University of Colorado, Boulder, Colorado
and
R. L. GRAHAM
Bell Telephone Laboratories, Inc., Murray Hill, New Jersey

For a sequence of integers $S = (s_1, s_2, \ldots)$, we denote by $P(S)$ the set

$$\left\{ \sum_{k=1}^{\infty} \epsilon_k s_k : \epsilon_k = 0 \text{ or } 1, \sum_{k=1}^{\infty} \epsilon_k < \infty \right\}.$$  

We say that $S$ is complete if all sufficiently large integers belong to $P(S)$. Conditions under which a sequence $S$ is complete have been studied by a number of authors. These sequences have ranged from the slowly growing sequences of Erdős [3] and Folkman [4] ($s_n = O(n^2)$), the polynomial and near-polynomial sequences of Roth and Szekeres [9], Graham [5] and Burr [1], to the near-exponential sequences of Cassels [2] ($s_n = O(n/\log n)$) and the exponential sequences of Lekkerkerker [7] and Graham [6] ($s_n = \lfloor 2e^n \rfloor$). In this note, we investigate sequences in which each term is a Fibonacci number, i.e., an integer $F_n$ defined by the linear recurrence

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0,$$

with $F_0 = 0, \ F_1 = 1$.

For a sequence $M = (m_1, m_2, \ldots)$ of nonnegative integers, let $S_M$ denote the nondecreasing sequence which contains precisely $m_k$ entries equal to $F_k$. It was noted in [7] that for $M = (1, 1, 1, \ldots)$, $S_M$ is complete but the deletion of any two terms of $S_M$ destroys the completeness. Further, it was shown in [1] that for any fixed $a$, if $M = (a, a, a, \ldots)$ then some finite set of entries can be deleted from $S_M$ so that the resulting sequence is not complete. This result can be strengthened as follows (where $\tau$ denotes $(1 + \sqrt{5})/2$).

249
Theorem 1. If

$$\sum_{k=1}^{\infty} m_k \tau^{-k} < \infty,$$

then some finite set of entries of $S_M$ can be deleted so that the resulting sequence is not complete.

Proof. The proof uses the ideas of Cassels [2]. Let $\|x\|$ denote $\min |x-n|$ where $n$ ranges over all integers. It is well known that $F_n$ can be explicitly written as

$$F_n = \frac{1}{\sqrt{5}} \left( \tau^n - (-\tau)^{-n} \right).$$

Thus

$$\sum_{s \in S_M} \|s\tau\| = \sum_{k=1}^{\infty} m_k \|F_k \tau\|$$

$$= \sum_{k=1}^{\infty} m_k \|F_k \tau - F_{k+1}\|$$

$$= \frac{1}{\sqrt{5}} \sum_{k=1}^{\infty} m_k \left\| \frac{(\tau^2 + 1)}{\tau} (-\tau)^{-k}\right\|$$

$$\leq \left| \frac{\tau^2 + 1}{\tau \sqrt{5}} \right| \sum_{k=1}^{\infty} m_k \tau^{-k} < \infty$$

by the hypothesis of the theorem. Hence, by deleting a sufficiently large initial segment of $S_M$, we can form a sequence $S^*_M$ for which
\[ \sum_{s \in S_M^*} \| s \tau \| < 1/4 . \]

But \( \tau \) is irrational so that for infinitely many integers \( m \), we have

\[ \| m \tau \| > 1/4. \]

The subadditivity of \( \| \cdot \| \) shows that such an \( m \) cannot belong to \( \mathcal{P}(S_M^*) \). This proves the theorem.

It follows in particular that if \( 1 < \theta < \tau \) and \( m_k = 0(\theta^k) \) then \( S_M \) is not "strongly complete," i.e., the deletion of some finite set of entries from \( S_M \) can result in a sequence which is not complete.

In the other direction, however, we have the following result.

**Theorem 2.** Suppose for some \( \epsilon > 0 \) and some \( k_0, \ m_k > \epsilon \tau^k \) for \( k > k_0 \). Then \( S_M \) is strongly complete.

**Proof.** For a fixed integer \( t \), let \( M_t \) denote the sequence

\[ (0, 0, \ldots, 0, m_{t+1}, m_{t+2}, \ldots) . \]

It is sufficient to show that \( S_{M_t} \) is complete. We recall the identity

(1) \[ F_{n+2k} + F_{n-2k} = L_{2k} F_n , \]

where \( L_r \) is the sequence of integers defined by \( L_{n+2} = L_{n+1} + L_n, \ n \geq 0 \), with \( L_0 = 2, \ L_1 = 1 \). It is easily shown that \( F_r \leq \tau^r \) and

\[ L_r \geq \frac{1}{2} \tau^r \]

for \( r \geq 0 \). We can assume without loss of generality that \( t > k_0 \) and \( \epsilon \tau^t > 2 \). Choose \( \ell > 4/\epsilon \) and \( n > t + 2\ell \). We can form sums of pairs \( F_{n+2k} + F_{n-2k} \) from \( S_{M_t} \) to get at least \( \epsilon \tau^{n-2k} \) copies of \( L_{2k} F_n \) (by (1)) for \( 0 \leq k \leq \ell \). Since \( \epsilon \tau^{n-2k} > \epsilon \tau^t > 2 \) then these sums can be used to form all the
multiples $uF_n$,

$$1 \leq u \leq \sum_{k=0}^{\ell} \varepsilon r^{n-2k} L_{2k}.$$ 

Since

$$L_r \geq \frac{1}{2} r^r,$$

then we have formed all multiples $uF_n$,

$$1 \leq u \leq \frac{\varepsilon (\ell + 1)}{2} r^n.$$

The same argument can be applied to the terms $F_{n+1+2k}$ (which are distinct from the terms previously considered) to form all multiples $vF_{n+1}$,

$$1 \leq v \leq \frac{\varepsilon (\ell + 1)}{2} r^{n+1}.$$ 

Of course, $F_n$ and $F_{n+1}$ are relatively prime so that the set of integers of the form $xF_n + yF_{n+1}$, $x$ and $y$ nonnegative integers, contains all integers $> F_n F_{n+1} - F_n - F_{n+1}$ (cf. [8]). For any integer

$$N_j = F_n F_{n+1} - F_n - F_{n+1} + j, \quad 1 \leq j \leq F_{n+2},$$

the coefficients $x_j$ and $y_j$ in a representation

$$N_j = x_j F_n + y_j F_{n+1}$$

certainly satisfy $x_j \leq F_{n+1}$, $y_j \leq F_n$. Thus, $x_j, y_j \leq \tau^{n+1} < 2\tau^n$. Since $u$ and $v$ can range up to

$$\frac{\varepsilon (\ell + 1)}{2} \tau^n > 2\tau^n$$
then by using the multiples of $F_n$ and $F_{n+1}$ we have just considered, we can represent all the $N_j$, $1 \leq j \leq F_{n+2}$, as elements of $P(S_{M'})$. Finally, since we have used at most $\epsilon \tau^{n-2}$ copies of $F_n$, $2 \leq i$, in this process, we still have available at least $\epsilon (\tau^{n+2} - \tau^{n-2}) > 1$ copies of $F_{n+i}$ to use in forming sums in $P(S_{M'})$. By adding sequentially a single copy of $F_{n+i}$, $i = 2, 3, 4, \cdots$, to the $N_j$, it is not difficult to see that all integers $\geq N_i$ belong to $P(S_{M'})$. Thus, $S_{M'}$ is complete and the theorem is proved.

It should be pointed out that the condition

$$\sum_{k=1}^{\infty} m_k \tau^{-k} = \infty$$

is not sufficient for the completeness of $S_M$ as can be seen from the example in which

$$m_k = \begin{cases} \left\lfloor \frac{k}{\tau} \right\rfloor & \text{if } k = 2^n \text{ for some } n \\ 0 & \text{otherwise} \end{cases}.$$ 

However, the proof of Theorem 2 directly applies to show that if $m_k/\tau^k$ is monotone and

$$\sum \frac{m_k}{\tau^k} = \infty$$

then $S_M$ is strongly complete.

It would be of interest to investigate refinements of these questions. Of course, similar results and questions arise for other $P-V$ numbers besides $\tau$ but we do not pursue these here.

REFERENCES


