ON UNIVERSAL GRAPHS

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INTRODUCTION

Many questions in extremal graph theory* take the following form: How many edges can a graph $G(n)$ with n vertices have and still contain as subgraphs *all* the graphs H belonging to a given class $\mathcal{H}$? Indeed, the most well-known theorem of this type is the theorem of Turán [20, 21] which asserts that if $\mathcal{H}$ is the class consisting of the single graph K_m, the complete graph on m vertices, then this maximum number of edges is given by,

$$\frac{m-2}{2(m-1)}(n^2 - r^2) + \binom{r}{2}$$

where r is the unique integer satisfying $r \equiv n \pmod{n-1}$ and $1 \leq r \leq m - 1$.

In general, let $t(\mathcal{H}; n)$ denote this maximum number of edges when the forbidden class is $\mathcal{H}$. In addition to Turán's theorem which deals with $t(\{K_m\}; n)$, numerous other results of this type are available, although usually only estimates as opposed to exact values for $t(\mathcal{H}; n)$ are available. For example:

(i) If C_{2m} denotes the cycle of length $2m$, then it has been shown by Erdös [9], and Bondy and Simonovits [3] that,

$$\frac{cn \log n}{\log \log n} < t(\{C_{2m}\}; n) < c_m n^{1 + 1/m}$$

for suitable positive constants c_1, c_m. In particular, it is known that $t(\{C_4\}; n) \sim \frac{1}{2} n^{3/2}$ [10, 14].

(ii) If $K_{3, 3}$ denotes the complete bipartite graph on two vertex sets of size 3, then Brown [4] and Kövari *et al.* [14] have proved that

$$cn^{5/3} < t(\{K_{3, 3}\}; n) \leq (2^{1/3} n^{5/3} + 3n)/2.$$

(iii) An attractive (and apparently difficult) conjecture of Erdös and Sós [9] asserts that for $\mathcal{H} = \mathcal{T}_m$, the class of all trees with m edges,

$$t(\mathcal{T}_m; n) = \left\lceil \frac{(m-1)n}{2} \right\rceil$$

Relatively little is currently known about this conjecture.

Recently a number of results have become available which deal the *complementary* extremal problem. That is, for a given class $\mathcal{H}$, what is the *least* number $s(\mathcal{H}; n)$ of edges a graph $G(n)$ on n vertices can have so that *all* $H \in \mathcal{H}$ occur as subgraphs of $G(n)$. It will also be of interest to consider the quantity $s(\mathcal{H})$ defined to be $\min\{s(\mathcal{H}; n): n = 1, 2, \ldots\}$. Such graphs containing all $H \in \mathcal{H}$ as subgraphs are

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* For undefined graph theory terminology, see [1] or [12].

sometimes said to be *universal* for the class $\mathcal{H}$. Since we are concerned with universal graphs with a minimum number of edges, this explains the title of this note.

We shall survey the results known (to us) concerning $s(\mathcal{H}; n)$ and $s(\mathcal{H})$ for various classes $\mathcal{H}$. In particular we describe a striking improvement recently found by us for $s(\mathcal{T}_m)$. We also mention numerous related open problems.

<center>BOUNDS FOR VARIOUS $s(\mathcal{H}; n)$</center>

Perhaps the first investigation of universal graphs was due to Rado and de Bruijn [18], who restricted their attention to infinite graphs. The first paper dealing with finite universal graphs was by Moon [15], who actually considered a somewhat different problem. Moon defined $\lambda(n)$ as the least integer N such that there is a graph with N vertices and having every graph on n vertices as an *induced* subgraph. He showed

$$2^{(n-1)/2} \le \lambda(n) \le \begin{cases} n2^{(n-1)/2}, & \text{for } n \text{ odd} \\ \dfrac{3n}{2\sqrt{2}} \cdot 2^{(n-1)/2}, & \text{for } n \text{ even} \end{cases}$$

As Moon points out in [16], the same bound also applies to the minimum number of vertices in a tournament (directed complete graph) containing all tournaments on n vertices as subgraphs.

One of the strongest results for an $s(\mathcal{H}; n)$ is due to Bondy [2]. It deals with the class $\mathcal{C}_n$ of all cycles of length $\le n$. Bondy has shown that

$$n + \log_2(n-1) - 1 < s(\mathcal{C}_n; n) < n + \log_2(n-1) + H(n) + O(1)$$

where $H(n)$ denotes $\min\{k: \overbrace{\log \log \cdots \log}^{k} n < 2\}$. It would be interesting to know which of the two bounds (if either!) is the "truth."

The most intensively studied class $\mathcal{H}$—from the point of view of $s(\mathcal{H})$—is the class $\mathcal{T}_n$ of all trees having n edges. The best estimate of $s(\mathcal{H})$ from below comes from the following simple observation. If $T \in \mathcal{T}_n$ is a subgraph of a graph G then the degree sequence $(d_1, d_2, \ldots, d_N)$ of G must dominate the degree sequence $(d_1', d_2', \ldots, d_{n+1}')$ of T, i.e., $d_k \ge d_k'$, $1 \le k \le n + 1$. Since for each k there is easily seen to be a tree $T(k) \in \mathcal{T}_n$ with degree sequence $(d_1(k), d_2(k), \ldots, d_{n+1}(k))$ satisfying $d_k(k) \ge \dfrac{n}{k}$ then the number of edges of G is at least,

$$\frac{1}{2}(d_1 + d_2 + \cdots + d_N) \ge \frac{1}{2}(d_1' + \cdots + d_{n+1}')$$

$$\ge \frac{1}{2} \sum_{k=1}^{n} \frac{n}{k} > \frac{1}{2} n \log n$$

That is, $$s(\mathcal{T}_n) > \tfrac{1}{2} n \log n. \tag{1}$$

The first upper bound for $s(\mathcal{T}_n)$ was given by Nebeský [17]. He proves various elementary facts about graphs G containing all $T \in \mathcal{T}_n$ as *spanning* subtrees and gives a construction for such G (which however has more than cn^2 edges for a fixed $c > 0$.) Nebeský's results were based, in part, on earlier related work of Sedláček [19].

The first subquadratic upper bound for $s(\mathcal{T}_n)$, which was presented by Chung and Graham [5], was

$$s(\mathcal{T}_n) < n^{1 + 1/\log \log n}$$

for n sufficiently large. This was subsequently strengthened by Chung et al. [6] to

$$s(\mathcal{T}_n) = O[n \log n(\log \log n)^2]$$

Very recently, we have finally succeeded in removing the $(\log \log n)^2$ term by proving that

$$s(\mathcal{T}_n) \le s(\mathcal{T}_n; n + 1) \le \frac{5}{\log 4} n \log n + O(n)$$

for n sufficiently large [7].

The proofs of all three results are rather different, the last one being, by far, the most complicated.

Almost nothing is known for the case $\mathcal{H} = \mathcal{G}_n$, the class of all graphs with n edges. The best bounds on $s(\mathcal{G}_n)$ currently available are essentially trivial:

$$c_1 n \log n < s(\mathcal{G}_n) < c_2 n^2$$

for suitable constants c_1 and c_2 [7]. It seems quite reasonable that we should have $s(\mathcal{G}_n) = o(n^{1 + \varepsilon})$ for any $\varepsilon > 0$, but unfortunately we cannot even prove $s(\mathcal{G}_n) = o(n^2)$.

A natural generalization of these questions occurs by restricting the class of containing graphs. For example, let us denote by $s_{\mathcal{T}}(\mathcal{T}_n)$ the minimum number of edges in any tree $T \in \mathcal{T}$ (the class of all trees) which contains all trees $T_n \in \mathcal{T}_n$ with n edges as subgraphs. Of course, we would expect $s_{\mathcal{T}}(\mathcal{T}_n)$ to be much larger than $s(\mathcal{T}_n)$. In fact, since $\mathcal{T}_n$ contains exponentially many trees and each subtree of a tree is an induced subgraph, it might be expected that $s_{\mathcal{T}}(\mathcal{T}_n) > c^n$ for some $c > 1$. Surprisingly, however, this is not the case. It has been shown by Chung et al. [6] that,

$$s_{\mathcal{T}}(\mathcal{T}_n) < \frac{2\sqrt{2}}{n} \exp \left[\frac{\log^2 n}{2 \log 2} \right] \tag{2}$$

In the other direction the only available bound is an annoyingly weak result [6],

$$s_{\mathcal{T}}(\mathcal{T}_n) > cn^2$$

for a suitable $c > 0$. We certainly suspect that $s_{\mathcal{T}}(\mathcal{T}_n)$ should grow annoyingly faster than any polynomial in n.

The inequality (2) is proved by means of considering the corresponding question for rooted trees. More precisely, let $s'_{\mathcal{T}}(\mathcal{T}_n)$ denote the minimum number of edges a rooted tree T' can have so that every rooted tree T'_n with n edges can be embedded in T' with the root of T'_n mapping onto the root of T'. Chung et al. [6] have shown that,

$$s_{\mathcal{T}}(\mathcal{T}_n) \le s'_{\mathcal{T}}(\mathcal{T}_n) \le s_{\mathcal{T}}(\mathcal{T}_n)(s_{\mathcal{T}}(\mathcal{T}_n) + 1)$$

Thus if $s_{\mathcal{T}}'(\mathcal{T}_n)$ grows faster than a polynomial, then so does $s_{\mathcal{T}}(\mathcal{T}_n)$. De Millo et al. [8] have recently considered the related case in which both the containing tree T' and the contained subtrees T'_n all have degrees bounded by a given constant d.

CONCLUDING REMARKS

Almost all the work in this topic remains to be done. We mention a few of the problems now. To begin with, is it true that $s(\mathcal{T}_n) = s(\mathcal{T}_n; n + 1)$? All values of $s(\mathcal{T}_n)$ which are known are in fact achieved by graphs with $n + 1$ vertices. We
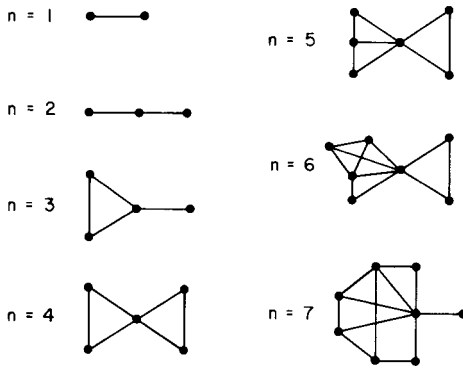
FIGURE 1

show graphs which achieve small values of $s(\mathcal{T}_n)$ and $s_{\mathcal{F}}(\mathcal{T}_n)$ in FIGURES 1 and 2, respectively. R. Read (unpublished) has suggested that the name *panderdron* be applied to such graphs (containing all small trees); Schuster has suggested the term *panarboreal* [22]. Is it possible that $s(\mathcal{T}_n) \sim \frac{1}{2}n \log n$?

As we have mentioned earlier, it would be highly desirable to show that for $\mathcal{G}_n$, the class of all graphs with n edges, $s(\mathcal{G}_n) = o(n^2)$. Of course with this case (as well as most of the others) we could require that the subgraphs be *induced*. With the exception of Moon's results [15], almost nothing is known here.
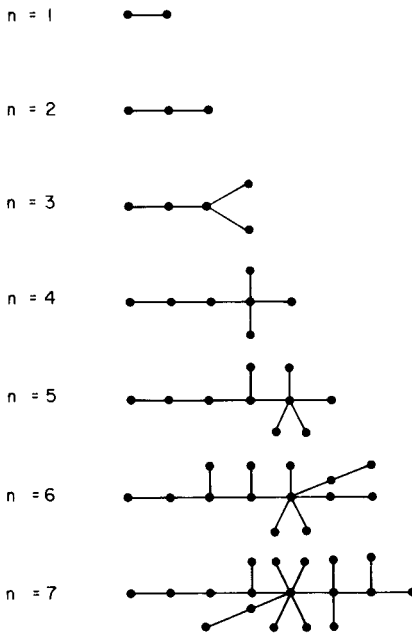

FIGURE 2

In the same spirit we might look at these questions for bipartite graphs, directed graphs, chromatic graphs and even hypergraphs. We suspect that for hypergraphs results might be substantially more difficult to obtain since, for example, even the analog of Turán's theorem for 3-uniform hypergraphs is not currently known [20, 21].

Finally we mention an interesting direction which has been taken by Howorka [13]. He defines $j(n)$ to be the minimum number of vertices a graph G can have so that every connected graph on at most n vertices can be embedded isometrically in G and asks for bounds on $j(n)$. Of course, similar questions can be asked when the graphs under consideration are measured by the number of edges instead of the number of vertices.

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