Euclidean Ramsey Theorems on the $n$-Sphere

R. L. Graham

BELL LABORATORIES, MURRAY HILL, NJ 07974

ABSTRACT

Let us call a finite subset $X$ of a Euclidean $m$-space $E^n$ Ramsey if for any positive integer $r$ there is an integer $n = n(X; r)$ such that in any partition of $E^n$ into $r$ classes $C_1, \ldots, C_r$, some $C_i$ contains a set $X'$ which is the image of $X$ under some Euclidean motion in $E^n$. Numerous results dealing with Ramsey sets have been proved in recent years although the basic problem of characterizing the Ramsey sets remains unsettled. The strongest constraints currently known are: (i) Any Ramsey set must lie on the surface of some sphere; (ii) Any subset of the set of vertices of a rectangular parallelepiped is Ramsey. In this paper we examine the corresponding problem in the case that our underlying spaces are (unit) $n$-spheres $S^n$ and the allowed motions are orthogonal transformations of $S^n$ onto itself. In particular, we show that for subsets of $S^n$ which are not too "large," results similar to (i) and (ii) hold.

1. INTRODUCTION

Let us call a finite subset $X$ of $E^n$ Ramsey if for any positive integer $r$ there is an integer $n = n(X; r)$ such that in any partition of $E^n = \bigcup_{k=1}^r C_k$, some $C_i$ contains a set $X'$ which is the image of $X$ under some Euclidean motion in $E^n$. Numerous results dealing with Ramsey sets in $E^n$ have been proved in recent years (e.g., see [1], [2], [3], [5], [6], [7], [9], [10], [11]), although the basic problem of characterizing the Ramsey sets remains unsettled. The best results currently known are the following. Let us call a set $Y \subseteq E^n$ spherical if it lies on the surface of some sphere in $E^n$, i.e., for some $\bar{z} \in E^n$, all the distances $d(\bar{z}, \bar{y})$, $\bar{y} \in Y$, are equal (where $d$ denotes Euclidean distance). Also, we call a set $Y \subseteq E^n$ a brick if it is the set of $2^n$ vertices of some rectangular parallelepiped in $E^n$.

Theorem ([1]).

(i) Every brick is Ramsey.

(ii) Every Ramsey set is spherical.
In this article we examine the analogous question for the case that our underlying spaces are (unit) $n$-spheres $S^n = \{(x_0, \ldots, x_n): \Sigma_{k=0}^n x_k^2 = 1\} \subseteq \mathbb{E}^{n+1}$ and the allowed motions are orthogonal transformations of $S^n$ onto itself. In this case the unavoidable sets will be termed "sphere-Ramsey." It will turn out that for sets $X \subseteq S^n$ which are not too large (in a sense to be made precise later), a result similar to the preceding Theorem holds. For the remaining cases, only very preliminary results are available, although we suspect that much more is very likely true.

2. NECESSARY CONDITIONS

**Theorem 1.** Let $X = \{\bar{x}_1, \ldots, \bar{x}_m\}$ be a set of points in $\mathbb{E}^n$ such that:

(i) for some nonempty $I \subseteq \{1,2,\ldots,m\} = [m]$, there exist nonzero $\alpha_i$, $i \in I$, such that

$$\sum_{i \in I} \alpha_i \bar{x}_i = \bar{0};$$

(ii) for all nonempty $J \subseteq I$,

$$\sum_{j \in J} \alpha_j \neq 0.$$

Then there exists $r = r(X)$ such that for any $N$, there is a partition $S_N = \bigcup_{k=1}^r C_k$ such that no $C_i$ contains a copy of $X$.

**Proof.** Consider the homogeneous linear equation

$$\sum_{i \in I} \alpha_i z_i = 0. \quad (*)$$

By (ii), Rado's theorem for the partition regularity of this equation over $\mathbb{R}^+$ (see [8] or [7]) implies that it is not regular, i.e., for some $r$ there is an $r$-coloring $\chi: \mathbb{R}^+ \rightarrow [r]$ such that (*) has no monochromatic solution. Color the points of $S_N = \{(x_0, \ldots, x_N) \in S^N: x_0 > 0\}$ by

$$\chi^*(\bar{x}) = \chi(\bar{u} \cdot \bar{x}),$$

where $\bar{u}$ denotes the unit vector $(1,0,0,\ldots,0)$. Thus, the color of $\bar{x} \in S^N$ just depends on its distance from the "north pole" of $S^N$.

For each nonempty subset $J \subseteq I$, consider the equation

$$\sum_{j \in J} \alpha_j z_j = 0. \quad (*)_J$$
Of course, by (ii) this also fails to satisfy the (necessary and sufficient) condition of Rado for partition regularity. Hence, there is a coloring \( \chi_f \) of \( \mathbb{R}^+ \) (using \( r_f \) colors) so that \((*)\) has no monochromatic (under \( \chi_f \)) solution. As before, we can color \( S^N_+ \) by giving \( \bar{x} \in S^N_+ \) the color

\[
\chi_f(\bar{x}) = \chi_f(\bar{x} \cdot \bar{u}).
\]

Now, form the product coloring \( \hat{\chi} \) of \( S^N_+ \) by defining for \( \bar{x} \in S^N_+ \),

\[
\hat{\chi}(\bar{x}) = (\chi_f(\bar{x}), \ldots, \chi_f(\bar{x}), \ldots)
\]

where the sequence has length \( 2|I| - 1 \) and the indices of the \( \chi_f \) range over all nonempty subsets \( J \subseteq I \). The number of colors required by the coloring \( \hat{\chi} \) is at most \( \Pi_{\theta \neq J \subseteq I} r_f \equiv R \).

An important property of \( \hat{\chi} \) is this. Suppose we extend \( \hat{\chi} \) to \( S^N \equiv \{ (x_0, \ldots, x_N) \in S^N : x_0 \geq 0 \} \) by assigning all \( R \) colors to any point in \( S^N \setminus S^N_+ \) i.e., having \( x_0 = 0 \). Then the only monochromatic solution to \((*)\) in \( \mathbb{R}^+ \cup \{0\} \) is \( z_i = 0 \) for all \( i \in I \).

Next, construct a similar coloring \( \tilde{\chi} \) on \( S^N = \{ -\bar{x} : \bar{x} \in S^N_+ \} \), but using \( R \) completely different colors. This assures that any set \( X \) which hits both hemispheres \( S^N_+ \) and \( S^N_- \) cannot be monochromatic. Finally, we have to color the equator

\[
S^{N-1} = \{ \bar{x} \in S^N : x_0 = 0 \}.
\]

By our construction, any copy of \( X \) which is not contained entirely in \( S^{N-1} \) cannot be monochromatic. Hence, it suffices to color \( S^{N-1} \) avoiding monochromatic copies of \( X \) where we may use any of the \( 2R \) colors previously used in the coloring of \( S^N_+ \cup S^N_- \). By induction, this can be done provided we can so color \( S^1 \). However, since \( m > 1 \), then \( S^1 \) can in fact always be 3-colored without a monochromatic copy of \( X \) (in fact, of any 2-element subset of \( X \) since the corresponding graph has maximum degree 2). This proves the theorem. ■

Note that if \( X \) is a constant distance \( d \neq 90^\circ \) from some point \( \bar{r} \in S^n \), then \( X \) cannot satisfy both (i) and (ii). For

\[
\sum_{i \in I} \alpha_i \bar{x}_i = \bar{0}
\]

implies

\[
0 = \bar{r} \cdot \left( \sum_{i \in I} \alpha_i \bar{x}_i \right) = \sum_{i \in I} \alpha_i \bar{r} \cdot \bar{x}_i = (\cos d) \cdot \sum_{i \in I} \alpha_i
\]
\[ \sum_{i \in I} \alpha_i = 0 \]

since \( \cos d \neq 0 \).

However, these are not the only sets not ruled out from being possible Ramsey sets by Theorem 1. Another such example is given by the 3-point set

\[ T = \left\{ (1,0), \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \left( -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \right\} = \{ t_1, t_2, t_3 \} \]

(corresponding to the three cube roots of unity). Their linear dependence is given by

\[ t_1 - t_2 - t_3 = 0 \]

which does not satisfy (ii).

We restate Theorem 1 in its positive form.

**Theorem 1.** If \( X \) is sphere-Ramsey, then for any linear dependence \( \sum_{i \in I} \alpha_i x_i = 0 \) there must exist a nonempty \( J \subseteq I \) such that \( \sum_{j \in J} \alpha_j = 0 \).

3. SUFFICIENT CONDITIONS—SMALL BRICKS

Let us call an \( m \)-dimensional brick with edge lengths \( \lambda_1, \lambda_2, \ldots, \lambda_m \) small if

\[ \sum_{i=1}^{m} \lambda_i^2 \leq 2. \]

**Theorem 2.** Every small brick is sphere-Ramsey.

**Proof.** We sketch the proof (which has the same basic structure as that of the Hales–Jewett theorem given in [6]). Let a fixed number \( r \) of colors be given. For \( m = 1 \), the theorem is immediate: we simply consider the \( r + 1 \) points

\[
\begin{align*}
& \{ (\beta_1, 0, 0, \ldots, 0, \gamma) \\
& (0, \beta_1, 0, \ldots, 0, \gamma) \\
& (0, 0, \beta_1, \ldots, 0, \gamma) \\
& \vdots \\
& (0, 0, 0, \ldots, \beta_1, \gamma)
\end{align*}
\]
where $\beta_1 = \lambda_1 / \sqrt{2} \leq 1$ and $\gamma^2 + \beta_1^2 = 1$. These $r + 1$ points are on $S^{r+1}$. Since they are $r$-colored, then some pair must have the same color. This pair has distance $\beta_1 \sqrt{2} = \lambda_1$, which is the desired conclusion.

In general, for a $\lambda_1 \times \ldots \times \lambda_m$ brick $B$, the set $S$ of points we consider is of the form

\[
\begin{array}{ccc}
N_m & N_{m-1} & \cdots & N_1 \\
(0, \ldots, \beta_m, \ldots, 0, 0, \ldots, \beta_{m-1}, \ldots, 0, 0, \ldots, 0, \beta_1, \ldots, 0, \gamma)
\end{array}
\]

That is, $S$ consists of $(N_m + N_{m-1} + \cdots + N_1 + 1)$-tuples in which exactly one of the entries in the $j$th block (of length $N_j$) is $\beta_j = \lambda_j / \sqrt{2}$ and all other entries are 0, with the exception of the last entry

\[
\gamma = \left(1 - \sum_{j=1}^{m} \beta_j^2\right)^{1/2},
\]

chosen so that all points are a unit $N$-sphere with $N = N_m + N_{m-1} + \cdots + N_1$. The hypothesis (1) guarantees that $\gamma$ is real. The key to this construction is (as usual) in the choice of the $N_j$’s. Needless to say, for the proof to work, they must grow very rapidly.

As an example, we consider the case $m = 2$. Choose $N_1 = r + 1$, $N_2 = r^{r+1} + 1$. An $r$-coloring $\chi$ of $S$ induces an $r^{r+1}$-coloring $\chi'$ of the set $S'$

\[
S' = \{(0, \ldots, \beta_2, \ldots, 0, \gamma)\}
\]

by

\[
\chi'(s'_1) = \chi'(s'_2), \quad s'_1, s'_2 \in S'
\]

iff

\[
\chi(s'_1 t) = \chi(s'_2 t)
\]

for all

\[
N_1
\]

$t \in \{(0, \ldots, \beta_1, \ldots, 0, \gamma)\} = T_1$

where the concatenation $s'_1 t$ has the obvious interpretation as an element of $S$. Since $|S'| = N_2 = r^{r+1} + 1$ and $S'$ is $r^{r+1}$-colored, then some pair of points $s'_1, s'_2 \in S'$ have $\chi'(s'_1) = \chi'(s'_2)$, i.e., $\chi(s'_1 t) = \chi(s'_2 t)$ for all $t \in T_1$. Since $\chi$ is an $r$-coloring and $|T_1| = N_1 = r + 1$, then some pair of points $t, t' \in T_1$ have

\[
\chi(s'_1 t) = \chi(s'_1 t').
\]
Of course, this implies
\[ \chi(s'_1t) = \chi(s'_2t) = \chi(s'_2t') = \chi(s'_2t'). \]

But
\[ d(s'_1t, s'_1t') = \beta_1\sqrt{2} = \lambda_1 = d(s'_2t, s'_2t') \]
\[ d(s'_1t, s'_2t) = \beta_2\sqrt{2} = \lambda_2 = d(s'_1t', s'_2t') \]
so that these 4 points form the desired monochromatic \( \lambda_1 \times \lambda_2 \) brick.

The general result follows by the same techniques where, in general, we choose \( N_i = r + 1 \) and \( N_{i+1} = 1 + r^{N_1N_2\cdots N_j} \) for \( j \geq 1 \). Specifically, we think of \( S \) as \( S(m) \times T(m) \), where \( S(m) \) consists of the \( N_{m} N_{m} \)-tuples \((0, \ldots, \beta_{m}, \ldots, 0)\) and \( T(m) \) consists of the \( N_1N_2 \cdots N_{m-1} \) complementary \((N_1 + \cdots + N_{m-1} + 1)\)-tuples

\[
\begin{array}{ccc}
N_{m-1} & N_{m-2} & N_1 \\
0, \ldots, \beta_{m-1}, \ldots, \beta_{m-2}, \ldots, \ldots, \ldots, \beta_1, \ldots, \gamma).
\end{array}
\]

The initial \( r \)-coloring \( \chi \) of \( S \) induces an \( r^{N_1\cdots N_{m-1}} \)-coloring \( \chi' \) of \( S(m) \) by
\[ \chi'(s'_1t') = \chi'(s'_2), \quad s'_1, s'_2 \in S(m) \]
iff
\[ \chi(s'_1t) = \chi(s'_2t) \quad \text{for all} \ t \in T(m). \]

Since
\[ |S(m)| = N_m = 1 + r^{N_1\cdots N_{m-1}}, \]
then there exists a pair of points, say \( s_1, s_2 \in S(m) \), such that
\[ \chi'(s'_1) = \chi'(s'_2). \]

Also, there is induced \( r \)-coloring \( \hat{\chi} \) of \( T(m) \) by
\[ \hat{\chi}(t) = \chi(s'_1t), \quad t \in T(m). \]

By induction, there is a monochromatic \( \lambda_1 \times \cdots \times \lambda_m \) brick under the coloring \( \hat{\chi} \) of \( T(m) \). By the definition of \( \hat{\chi} \) and \( \chi' \), this extends to a monochromatic \( \lambda_1 \times \cdots \times \lambda_m \) brick in the original coloring of \( S \).
By suitable manipulations, it can be shown that the $N_m$ satisfy
\[
N_m \leq (r+2)^2 (r+2) = (r+2)^{r+1}.
\]

**Large Bricks.** Bricks which have a main diagonal of length exceeding 2 seem much less tractable, although we expect that any $\lambda_1 \times \cdots \times \lambda_m$ brick with
\[
\lambda_1^2 + \cdots + \lambda_m^2 < 4
\]
is sphere-Ramsey. We can only prove this in the case $m = 1$.

**Theorem 3.** Let $B$ be the set $\{-\lambda/2, \lambda/2\}$ where $0 < \lambda < 1$. Then $B$ is sphere-Ramsey.

**Proof.** It is enough to show that the graph $G_n$ with vertex set $S^n$ and edge set $\{[x, y]: d(x, y) = \lambda\}$ has chromatic number tending to infinity as $n$ tends to infinity. To prove this, we use the following recent result of Frankl and Wilson (which was suggested by I. Bárány, Z. Füredi, and J. Pach).

**Theorem [4]:** Let $\mathcal{F}$ be a family of $k$-sets of $[n]$ such that for some prime power $q$,
\[
|F \cap F'| \not\equiv k \pmod{q}
\]
for all $F \neq F'$ in $\mathcal{F}$. Then
\[
|\mathcal{F}| \leq \binom{n}{q-1}.
\]

For a fixed $r$, choose a prime power $q$ so that
\[
\binom{2(1+\varepsilon)q}{(1+\varepsilon)q} > r \binom{2(1+\varepsilon)q}{q-1},
\]
where $\lambda = 2\beta\sqrt{2q}$, and $\varepsilon > 0$ and $\alpha$ are chosen so that
\[
\alpha^2 + 2(1+\varepsilon)q \beta^2 = 1.
\]
and \( N = (1 + \epsilon)q \) is an integer. Consider the set

\[
S = \left\{ (s_0, \ldots, s_{2N}) : s_0 = \alpha, s_i = \pm \beta, \sum_{i=1}^{2N} s_i = 0 \right\}.
\]

To each \( s \in S \) associate the subset

\[
F(s) = \{ i \in [2N] : s_i = \beta \}.
\]

Thus, the family

\[
\mathcal{F} = \{ F(s) : s \in S \}
\]

consists of the \( \binom{2N}{N} \) \( N \)-element subsets of \([2N]\). If \( F, F' \in \mathcal{F}, F \neq F' \), then

\[
| F \cap F' | \equiv N \pmod{q}
\]

iff

\[
| F \cap F' | = N - q = \epsilon q.
\]

If the elements of \( \mathcal{F} \) are \( r \)-colored, then some color class must contain at least

\[
\frac{1}{r} | \mathcal{F} | = \frac{1}{r} \binom{2N}{N} > \binom{2N}{q-1}
\]

elements of \( \mathcal{F} \). However, by Frankl–Wilson, if \( | F \cap F' | = \epsilon q \) never occurs, then the number of \( N \)-sets must be at most \( \binom{2N}{q-1} \), which is a contradiction. Thus, some monochromatic pair \( F, F' \) must have

\[
| F \cap F' | = \epsilon q.
\]

This means that the corresponding points \( s, s' \in S \) must (up to a permutation of coordinate positions) look like

\[
s = (\alpha, \beta, \ldots, \beta, \beta, \ldots, \beta, -\beta, \ldots, -\beta, -\beta, \ldots, -\beta),
\]

\[
s' = (\alpha, \beta, \ldots, \beta, -\beta, \ldots, -\beta, \beta, \ldots, \beta, -\beta, \ldots, -\beta).
\]
Note that
\[ d(s, s') = \sqrt{8q \beta^2} = \lambda \]
and
\[ d(s, 0) = d(s', 0) = \alpha^2 + 2(1 + \varepsilon)q \beta^2 = 1, \]
i.e., \( s, s' \in S^{2N} \). This proves the theorem. □

As remarked previously, one would expect that the corresponding result should hold for any \( \lambda_1 \times \cdots \times \lambda_m \) brick provided \( \lambda_1^2 + \cdots + \lambda_m^2 < 4 \). However, we are unable to prove this for even the case \( m = 2 \).

4. SOME REMARKS ON EDGE COLORINGS

Instead of coloring the points of \( \mathbb{E}^n \), we could color the line segments in \( \mathbb{E}^n \) and, as before, look for monochromatic copies of some fixed structure \( C \) (again, up to some Euclidean motion). A set \( C \) of line segments which must always occur monochromatically in an \( r \)-coloring of \( \mathbb{E}^n \), provided only that \( n \) is sufficiently large as a function of \( r \) (and \( C \), \( r = 1, 2, 3, \ldots \), is said to be line-Ramsey. Several results on line-Ramsey sets were mentioned in [1], such as the fact that any line-Ramsey set must have all edges the same length (which we can assume is 1).

For a configuration \( C \) of unit line segments \( L_i \), let \( V(C) \) denote the set of endpoints of the \( L_i \). Form a graph \( G(C) \) with vertex set \( V(C) \) and having all the \( L_i \) as its edges.

Theorem 4. Suppose \( C \) is a configuration of unit line segments such that:

(i) \( V(C) \) is not spherical;
(ii) \( G(C) \) is not bipartite.

Then \( C \) is not line-Ramsey.

Proof. Since \( V(C) \) is not spherical, then by the previously mentioned necessary condition for \( V(C) \) to be Ramsey, there exists an \( r \) and, for each \( N \), an \( r \)-coloring \( \chi^r \) of \( \mathbb{E}^n \) so that \( V(C) \) does not occur monochromatically. Let us color the unit line-segments \( \{x, y\} \) of \( \mathbb{E}^n \) by \( \chi^r(\{x, y\}) = \{\chi(x), \chi(y)\} \).
Consider a fixed copy $C'$ of $C$. Since $V(C')$ is not monochromatic, there are two points of $V(C')$, say $x'$ and $y'$ with $\chi(x') \neq \chi(y')$. Suppose $C'$ is monochromatic under $\chi^*$. Then all edges of $G(C')$ must have color $\{\chi(x'), \chi(y')\}$ since both colors $\chi(x')$ and $\chi(y')$ occur in the coloring $V(C')$. By (ii), $G(C')$ is not bipartite, and so, contains an odd cycle. However, it is easy to see that this results in a contradiction since an odd cycle cannot have all its edges with color $\{\chi(x'), \chi(y')\}$. ■

By the same technique, we can show that if $V(C)$ does not lie on two concentric spheres then $C$ cannot be line-Ramsey, even when $G(C)$ is bipartite.

References


