A NEW RESULT ON COMMA-FREE CODES OF EVEN WORD-LENGTH

BETTY TANG, SOLOMON W. GOLOMB AND RONALD L. GRAHAM

1. Introduction. Comma-free codes were first introduced in [1] in 1957 as a possible genetic coding scheme for protein synthesis. The general mathematical setting of such codes was presented in [3], and the biochemical and mathematical aspects of the problem were later summarized and extended in [4].

Using the notation of [3], a set $D$ of $k$-tuples or $k$-letter words, $(a_1a_2\ldots a_k)$, where

$$a_i \in \mathbb{Z}_n = \{0, 1, 2, \ldots, n - 1\},$$

for fixed positive integers $k$ and $n$, is said to be a comma-free dictionary if and only if, whenever $(a_1a_2\ldots a_k)$ and $(b_1b_2\ldots b_k)$ are in $D$, the "overlaps"

$$(a_1a_{i+1}\ldots a_kb_1\ldots b_{i-1}), \quad 2 \leq i \leq k,$$

are not in $D$. This precludes codewords having a subperiod less than $k$; and two codewords which are cyclic permutations of one another cannot both be in $D$. Therefore at most one member from the non-periodic cyclic equivalence class of $(a_1\ldots a_k)$, i.e., from the set

$$\{(a_j\ldots a_k a_1\ldots a_{j-1}) | 1 \leq j \leq k \},$$

can be in $D$. The maximum number of codewords, $W_k(n)$, in the comma-free dictionary $D$ therefore cannot exceed the number of non-periodic cyclic equivalence classes of sequences of length $k$ formed from an alphabet of $n$ letters. Denoting the latter number by $B_k(n)$, we have formally,

$$W_k(n) \leq B_k(n)$$

where

$$B_k(n) = \frac{1}{k} \sum_{d|k} \mu(d)n^{k/d}$$

The summation is extended over all divisors $d$ of $k$, and $\mu(d)$ is the Möbius function.

Received November 21, 1983, and in revised form February 12, 1986. This research was supported in part by the National Security Agency, under Contract No. MDA904-83-H-0004.
Golomb, Gordon and Welch [3] proved that $W_k(n)$ attains the upper bound $B_k(n)$ for arbitrary $n$ if $k = 1, 3, 5, 7, 9, 11, 13, 15$, and conjectured that this is indeed the case for all odd $k$. The conjecture was proved by Eastman [2], who gave a construction for the maximal comma-free dictionaries. A simpler construction for these dictionaries was found by Scholtz [6].

The results for even integers $k$ were less complete. Golomb, Gordon and Welch [3] were able to prove that $W_k(n)$ cannot attain the bound $B_k(n)$ for $n > 3^{k/2}$; and in particular,

$$W_2(n) = \left[\frac{n^2}{3}\right]$$

where $[x]$ is the integral part of $x$, whereas

$$B_2(n) = \frac{n^2 - n}{2}.$$

It was also mentioned that for $k = 4$, we in fact have $W_4(n) < B_4(n)$ if $n \geq 5$, while $W_4(n) = B_4(n)$ if $n = 1, 2, 3$. The case for $n = 4$ was later solved in [5] by exhaustive computer search, which found $W_4(4) = 57 < B_4(4) = 60$.

An improvement on the relation between $k$ and $n$ such that $W_k(n) < B_k(n)$ for even $k$ was given by Jiggs [5]:

$$W_k(n) < B_k(n) \text{ if } n > 2^{k/2} + \frac{k}{2}.$$ 

We present a further improvement based on Jiggs' proof, which in turn gives rise to a very interesting combinatorial problem. We first present Jiggs' result (attributed by Jiggs to R. I. Jewett) with some modifications of notation.

We consider only the simpler problem of forming a comma-free dictionary $D$ with $\binom{n}{2}$ codewords of length $k = 2l$, with one representative from each cyclic class of the type $(a00\ldots0b00\ldots0)$, with $0 \leq a < b \leq n - 1$ and $l - 1$ 0's between $a$ and $b$. Clearly if these $\binom{n}{2}$ classes cannot be simultaneously represented in a comma-free dictionary, the full set of $B_k(n)$ classes cannot be so represented.

A half-word in $D$ is an $l$-tuple which is either the initial half or final half of some word in $D$. For each $d \in \mathbb{Z}_n$ and $1 \leq r \leq k/2$, let $u(d, r)$ denote the half-word with $d$ at the $r$-th position and 0 everywhere else. We assign a sequence

$$x^d = x_1^d x_2^d \ldots x_l^d$$

to each $d \in \mathbb{Z}_n$ where $x_r^d$ is defined in the following way:
\[ x_r^d = \begin{cases} 
2 & \text{if } u(d, r) \text{ is both initial and final} \\
1 & \text{if } u(d, r) \text{ is final only} \\
0 & \text{if } u(d, r) \text{ is initial only} \\
* & \text{if } u(d, r) \text{ is neither initial nor final.} 
\end{cases} \]

Jiggs showed that the sequences \( x^d \) have the following two properties:

1. If \( d \neq b \), then \( x_r^d \) and \( x_r^b \) cannot both be 2, for any \( 1 \leq r \leq l \). Thus at most \( l \) of the sequences \( x^d \) can contain the symbol 2.

2. Among the sequences in which the symbol 2 does not occur, if \( d \neq b \), there exists \( 1 \leq r \leq l \) such that either \( x_r^d = 0 \) and \( x_r^b = 1 \), or \( x_r^b = 0 \) and \( x_r^d = 1 \). (In particular, distinct letters of the alphabet must have distinct sequences.)

We call two sequences, \( x^d \) and \( x^b \), composed of 0, 1, and *, comparable if they have property (2). The two properties imply that the maximum number of distinct sequences \( x^d \) containing a 2 is \( l \), and the maximum number of distinct sequences \( x^d \) containing no 2 is \( 2^l \). Hence if \( |D| = B_k(n) \), then \( n \leq 2^{k/2} + k/2 \).

Our improvement on Jiggs' result is a consequence of the following observation.

**Theorem 1.1.** If \( d \neq b \) and \( r \neq s \), we cannot have both \( x_r^d = x_s^b = 1 \) and \( x_r^b = x_s^d = 0 \).

**Proof.** Suppose there exist \( r \neq s \) such that \( x_r^d = x_s^b = 1 \) and \( x_r^b = x_s^d = 0 \). Then we will have words of the following form:

\[
\begin{align*}
w_1 &= (0 \ldots 0p0 \ldots 0d0 \ldots 0), \\
w_2 &= (0 \ldots 0b0 \ldots 0q0 \ldots 0),
\end{align*}
\]

where the non-zero letters appear at positions \( r \) and \( l + r \), and

\[
\begin{align*}
w_3 &= (0 \ldots 0x0 \ldots 0b0 \ldots 0), \\
w_4 &= (0 \ldots 0d0 \ldots 0y0 \ldots 0),
\end{align*}
\]

where the non-zero letters appear at positions \( s \) and \( s + l \). The overlaps of \( w_1w_2 \) and \( w_3w_4 \) therefore contain all members of the cyclic equivalence class of \( (0 \ldots 0b0 \ldots 0d0 \ldots 0) \) and so \( D \) cannot contain a representative of this class and still be comma-free.

We will call two sequences \( x^d \) and \( x^b \) compatible if they satisfy the exclusion condition in Theorem 1.1. We will now address the combinatorial problem of determining the maximum size of a set \( S \) of sequences of length \( l \), composed of *, 0, and 1 such that the sequences are pairwise comparable and compatible.

**2. The minimal array.** Let \( t = t(l) \) be the maximum number of distinct \( l \)-tuples of 0's, 1's, and *'s which are pairwise comparable and compatible.
We will try to determine \( t \) indirectly. Suppose we have an array of empty boxes with \( t \) rows in the array. We must fill in each empty box with either *, 0 or 1 such that every two rows, taken as sequences, are comparable and compatible. We want to know the minimum number of distinct columns in the array when there are \( t \) rows. Let \( f(t) \) be that minimum number, and call the array thus obtained the minimum array \( M_t \). Obviously, \( f(t) \leq l \).

We define \( t(1) = 0 \). The value of \( f(t) \) for small \( t \) can be obtained without much difficulty. (See Table 1).

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 2, \ f(t) = 1 )</td>
</tr>
<tr>
<td>( M_2 = \begin{bmatrix} 0 \ 1 \end{bmatrix} )</td>
</tr>
<tr>
<td>( t = 3, \ f(t) = 2 )</td>
</tr>
<tr>
<td>( M_3 = \begin{bmatrix} 0 &amp; 0 \ 0 &amp; 1 \ 1 &amp; 1 \end{bmatrix} )</td>
</tr>
<tr>
<td>( t = 4, \ f(t) = 3 )</td>
</tr>
</tbody>
</table>

Note that there can be more than one minimal array \( M_t \) for each \( t \). Also, \( t \) as a function of \( l \) is simply the largest number \( s \) such that \( f(s) = l \). From Table 1 we get the values of \( t(l) \) for some \( l \). (See Table 2.)

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l )</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

We can immediately establish a few properties of \( f(t) \).

**Theorem 2.1.** \( f(t) \) is a monotonically non-decreasing function of \( t \).

**Proof.** Let \( s > t \). We can remove any \( s - t \) rows from the minimal array \( M_s \) and the remaining array of \( t \) sequences will still be pairwise comparable and compatible. Therefore \( f(t) \leq f(s) \).
THEOREM 2.2. $f(t + 1) \leq f(t) + 1$.

Proof. From the minimal array $M_t$, construct a set of $t + 1$ sequences and $f(t) + 1$ columns in the following way. A 1 is added to the end of every sequence in $M_t$, and a sequence $x^{t+1}$ of length $f(t) + 1$ containing all 0's is adjoined to the set. The sequences in the new set are still pairwise comparable and compatible, and so

$$f(t + 1) \leq f(t) + 1.$$ 

THEOREM 2.3. $f(t) \leq t - 1$.

Proof. The sequences in the following array of $t - 1$ columns are pairwise comparable and compatible, so $f(t) \leq t - 1$.

$$
\begin{array}{ccccccc}
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 1 & * \\
0 & 0 & 0 & \ldots & 1 & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & \ldots & * & * & * \\
0 & 1 & * & \ldots & * & * & * \\
1 & * & * & \ldots & * & * & * \\
\end{array}
$$

We now require the minimal array $M_t$ to be such that the number of "comparison sites" between every two sequences is as small as possible. In other words, if $x^d_r = x^d_s = 0$ and $x^b_r = x^b_s = 1$ for some $r \neq s, 1 \leq r, s \leq f(t)$, we will replace either $x^d_r$ or $x^b_s$, or both, by * so long as the resulting array is still pairwise comparable and compatible.

LEMMA 2.4. In a minimal array $M_t$, there exists some column which contains *.

Proof. If the first column contains *, we are done. If not, we can assume that $x^d_1 = 0, d = 1, \ldots, s$, and $x^d_1 = 1, d = s + 1, \ldots, t$. Let $1 < r \leq f(t)$ and consider the $r$-th column. If again $x^d_r = 0, d = 1, \ldots, s$, and $x^d_r = 1, d = s + 1, \ldots, t$, we can eliminate the $r$-th column and the resulting array is still pairwise comparable and compatible, and therefore $M_t$ is not a minimal array. Suppose $x^d_r = 1$ for some $l \leq d \leq s$; then $x^d_r$ must be either * or 1 for all $s + 1 \leq d \leq t$ or else we will have non-compatible sequences. Since the number of comparison sites between every two sequences has to be minimum, all the $x^d_r$s, $s + 1 \leq d \leq t$, in fact have to be * because comparison sites already occur at the first column. The situation is similar if $x^d_r = 0$ for some $s + 1 \leq d \leq t$.

THEOREM 2.5. $f(2t) > f(t)$. 
Proof. We prove this by induction. \( f(2) = 1 > 0 = f(1) \). Assume \( f(t - 1) < f(2t - 2) \), but \( f(2t) = f(t) \). From Theorems 2.1 and 2.2, we must have
\[
f(2t - 2) \leq f(t - 1) + 1
\]
and
\[
f(2t) = f(t) \leq f(t - 1) + 1,
\]
and therefore
\[
f(s) = f(t - 1) + 1, \ t \leq s \leq 2t.
\]
In particular,
\[
f(t + 1) = f(t - 1) + 1.
\]
Consider the minimal array \( M_{2t} \), and suppose the \( r \)-th column contains at least one *\). The total number of entries which are not * in this column therefore cannot be more than \( 2t - 1 \). Without loss of generality, assume the number of 0's in this column is less than or equal to \( t - 1 \). If we now remove all rows in \( M_{2t} \), with 0 at the \( r \)-th position and also remove the \( r \)-th column, the resulting array has at least \( t + 1 \) rows and \( f(t) - 1 \) columns since \( f(2t) = f(t) \). The \( t + 1 \) rows are still pairwise comparable and compatible, whence \( f(t + 1) \leq f(t - 1) \), contradicting
\[
f(t + 1) = f(t - 1) + 1.
\]
We can now make a rough estimate of \( f(t) \). From Table 1 and Theorem 2.5, the best lower bound we can get is
\[
f(6 \cdot 2^i) \geq 4 + i, \ i = 0, 1, 2, \ldots
\]
Using the substitution \( t = 6 \cdot 2^i \), we get
\[
f(t) \geq q(t), \ t \geq 6,
\]
where
\[
q(t) = 4 + \frac{\log t - \log 6}{\log 2}
\]
which gives
\[
t(t) \leq 3 \cdot 2^{t - 3}.
\]

3. A graph structure on the minimum array. Given a minimal array \( M_t \), define a graph \( G_s \), for each \( 1 \leq s \leq f(t) \), on the vertex set \( V = \{1, 2, \ldots, t\} \) by assigning an edge between vertices \( b \) and \( d \), \( b \neq d \), if and only if either \( x_s^b = 0 \) and \( x_s^d = 1 \), or \( x_s^b = 1 \) and \( x_s^d = 0 \). Let
\[
A_s = \{b|1 \leq b \leq t, x_s^b = 0\}
and

\[ B_s = \{ b | 1 \leq b \leq t, x_s^b = 1 \}. \]

\( G_s \) is then a complete bipartite graph on the vertex sets \( A_s \) and \( B_s \), and is non-empty by comparability and the minimality of \( M_r \). We have the following observation.

**Lemma 3.1.** There do not exist \( s \) and \( s' \), \( 1 \leq s, s' \leq f(t) \), such that both

\[ A_s \cap B_{s'} \neq \emptyset \quad \text{and} \quad A_{s'} \cap B_s \neq \emptyset. \]

**Proof.** Suppose there exist \( b \) and \( d \) such that

\[ b \in A_s \cap B_{s'} \quad \text{and} \quad d \in A_{s'} \cap B_s. \]

Then

\[ x_s^b = x_{s'}^d = 0 \quad \text{and} \quad x_s^b = x_s^d = 1, \]

which implies \( x^b \) and \( x^d \) are not compatible sequences.

Now construct a graph \( G \) on the vertex set \( V = \{ 1, 2, \ldots, t \} \) by assigning an edge between \( b \) and \( d \) if and only if \( x^b \) and \( x^d \) are comparable sequences. Since all the \( x^b \)'s, \( 1 \leq b \leq t \), are pairwise comparable, \( G \) is a complete graph on \( V \). Moreover, the \( G_s \)'s, \( 1 \leq s \leq f(t) \), are a minimal cover of \( G \), that is,

\[ G = \bigcup_{s=1}^{f(t)} G_s \]

since every edge in \( G \) is also an edge in some \( G_s \), and \( f(t) \) is the minimum number of columns in \( M_r \).

Let \( \lambda_s = |A_s| \cdot |B_s| \), which gives the number of edges in the graph \( G_s \). Suppose

\[ \lambda = \lambda(t) = \max_{1 \leq s \leq f(t)} \lambda_s. \]

**Lemma 3.2.** \( f(t) \geq \binom{t}{2}/\lambda \), where \( \binom{t}{2} \) is the binomial coefficient.

**Proof.** Since \( G \) is a complete graph on a set of \( t \) vertices, there are \( \binom{t}{2} \) edges in \( G \). The minimal covering of \( G \) by all the \( G_s \)'s implies

\[ \binom{t}{2} \leq \sum_{s=1}^{f(t)} \lambda_s < \lambda f(t). \]

**Lemma 3.3.** There does not exist \( 1 \leq s \leq f(t) \) such that \( G_s \) has an edge between two vertices in both \( A_s \) and \( B_s \) for all \( 1 \leq s' \leq f(t) \).
Proof. If $G_s$ has an edge in $A_s'$ and $B_s$, then there exist $b_1, b_2, d_1, d_2$ such that $b_1, d_1 \in A_s'$ with $b_1 \in A_s$ and $d_1 \in B_s$ and $b_2, d_2 \in B_s'$ with $b_2 \in A_s$ and $d_2 \in B_s$. This implies $A_s' \cap B_s \neq \emptyset$ and $A_s \cap B_s' \neq \emptyset$.

In particular, let $s' = r$ where $\lambda = \lambda_r$ and assume without loss of generality that $|A_r| \geq |B_r|$. Lemma 3.3 asserts that in the complete graph $G$, the edges between vertices in $A_r$ and those in $B_r$ are covered separately. We therefore have

**Lemma 3.4.** $f(t) \geq f(|A_r|) + f(|B_r|)$.

So far $f$ is a function defined on the positive integers only. For convenience sake, extend $f$ to a function $\tilde{f}$ defined on all nonnegative real numbers by the following:

$$
\tilde{f}(t) = \begin{cases} 
  f(t) & \text{if } t \text{ is an integer} \\
  f(\lceil t \rceil) & \text{if } t \text{ is not an integer}
\end{cases}
$$

where $\lceil t \rceil$ is the smallest integer larger than or equal to $t$. Henceforth we will refer to $\tilde{f}(t)$ as a function defined on all $t \in [0, \infty)$ when we really mean $f(t)$.

**Lemma 3.5.** $f(t) \geq f(\sqrt{\lambda}) + f\left(\frac{\lambda}{t}\right)$.

**Proof.** We have

$$
\lambda = \lambda_r = |A_r| \cdot |B_r| \leq |A_r|^2
$$

or $|A_r| \geq \sqrt{\lambda}$. Moreover,

$$
|B_r| = \frac{\lambda}{|A_r|} \geq \frac{\lambda}{t}
$$

We then have, from the last lemma and the monotonicity of $f$,

$$
\tilde{f}(t) \geq f(\sqrt{\lambda}) + f\left(\frac{\lambda}{t}\right).
$$

**Corollary 3.6.** $\tilde{f}(t) \geq \max\left(\frac{t(t-1)}{2}, f(\sqrt{\lambda}) + f\left(\frac{\lambda}{t}\right)\right)$.

This additional property of $\tilde{f}(t)$ helps establish a larger lower bound for it.

**Theorem 3.7.** There exists a constant $0 < c_0 < 1$ such that

$$
\tilde{f}(t) \geq \exp\sqrt{c_0 \log(t)} \text{ for } t \geq a > 0.
$$

**Note.** We prove the theorem by actually taking $c_0 = 0.71$. It can be shown that
\( q(t) \geq \exp \sqrt{0.71 \log(t)} \) for \( 6 \leq t \leq T_0 \),

where \( q(t) \) is the bound in the last section and \( T_0 = 208,562 \) is the largest integer \( t \) such that

\( q(t) \geq \exp \sqrt{0.71 \log(t)} \)

and hence

\( f(t) \geq \exp \sqrt{0.71 \log(t)} \) for \( 6 \leq t \leq T_0 \).

**Proof of Theorem 3.7.** We proceed by induction using Corollary 3.6. All we need show is

\( f(t) \geq \exp \sqrt{0.71 \log(t)} \) for \( t \geq T_0 + 1 \).

Assume

\( f(s) \geq \exp \sqrt{0.71 \log(s)} \)

for all \( s \leq t - 1 \) where \( t \geq T_0 + 1 \). If

\( \frac{t(t - 1)}{2\lambda(t)} \geq \exp \sqrt{0.71 \log(t)} \),

we are done. Otherwise

\( \lambda(t) > \frac{t(t - 1)}{2\exp \sqrt{0.71 \log(t)}} \),

and hence

\[
\begin{align*}
    f(\sqrt{\lambda(t)}) + f\left(\frac{\lambda(t)}{t}\right) &\geq f\left(\frac{t(t - 1)}{2\exp \sqrt{0.71 \log(t)}}\right) \\
    &\quad + f\left(\frac{t - 1}{2\exp \sqrt{0.71 \log(t)}}\right).
\end{align*}
\]

For convenience, let \( u = \exp \sqrt{c_0 \log(t)} \) where \( c_0 = 0.71 \) and

\( G(t) = f\left(\frac{t(t - 1)}{2u}\right) + f\left(\frac{t - 1}{2u}\right) \).

Also, let

\( g(t) = \frac{t(t - 1)}{2u} \) and \( h(t) = \frac{g(t)}{t} \).

Simple calculus shows that both \( g(t) \) and \( h(t) \) are increasing functions, in particular for \( t \geq 6 \). Moreover, we must have

\( 6 < \sqrt{g(t)} < t - 1 \) and \( 6 < h(t) < t - 1 \).
By the induction hypothesis,

\[ G(t) \geq \exp \sqrt{c_0 \log \sqrt{g(t)}} + \exp \sqrt{c_0 \log h(t)}, \]

\[ = \exp \left( c_0 \log t + \frac{c_0}{2} \beta(t) \right)^{1/2} + \exp(c_0 \log t + c_0 \beta(t))^{1/2}, \]

where

\[ \beta(t) = \log \frac{t - 1}{2t} - \log u. \]

Note that \((t - 1)/2t\) is an increasing function of \(t\), and larger than \(1/e\) for \(t \geq T_0\). Hence

\[ \beta(t) > -1 - \log u \quad \text{for } t \geq T_0 + 1, \]

and therefore

\[ G(t) \geq \exp \left( c_0 \log t - \frac{c_0}{2} (1 + \log u) \right)^{1/2} \]

\[ + \exp(c_0 \log t - c_0(1 + \log u))^{1/2} \]

\[ = \exp \left[ \log u \left( 1 - \frac{c_0}{2(\log u)^2} (1 + \log u) \right)^{1/2} \right] \]

\[ + \exp \left[ \log u \left( 1 - \frac{c_0}{(\log u)^2} (1 + \log u) \right)^{1/2} \right]. \]

Since

\[ \frac{c_0}{(\log u)^2} (1 + \log u) < 1, \]

\[ \frac{1}{u} G(t) \geq z^2 + z \]

where

\[ z = z(t) = \exp \left[ -\frac{c_0}{2} \left( 1 + \frac{1}{\log u} \right) \right]. \]

Note that for \(t \geq T_0 + 1,\)

\[ z \geq \exp \left[ -\frac{c_0}{2} \left( 1 + \frac{1}{\sqrt{c_0 \log(T_0 + 1)}} \right) \right] > \frac{\sqrt{5} - 1}{2} \]

and therefore \(z^2 + z > 1\). Hence \(G(t) > u\), or

\[ f(t) > \exp \sqrt{0.71 \log(t)} \]
for \( t > T_0 \) also.

The constant \( c_0 = 0.71 \) is almost the best possible value, as \( T_0(0.72) = 132, 284 \), and in this case
\[
z(T_0) < \frac{\sqrt{5} - 1}{2}.
\]

With
\[
l \geq f(t) \geq \exp \sqrt{0.71 \log(t)},
\]
we get
\[
t(l) \leq l^{\log l/0.71},
\]
and a comma-free dictionary will not have the maximum size \( B_k(n) \) if
\[
n > \left( \frac{k}{2} \right)^{(\log k/2)/0.71} + \frac{k}{2}, \quad k \geq 8.
\]

Table 3 compares Jiggs' bound and the new bound on \( n \). Asymptotically, the new lower bound for \( n \) is significantly smaller. However, we suspect that compatibility is so strong a constraint that the bound on \( n \) could be dramatically reduced, probably to a polynomial in \( k \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>Jiggs' bound ( 2^{k/2} + k/2 )</th>
<th>New bound ( \lfloor (k/2)\exp(\log(k/2)/0.71) + k/2 \rfloor )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>20</td>
<td>18</td>
</tr>
<tr>
<td>10</td>
<td>37</td>
<td>43</td>
</tr>
<tr>
<td>20</td>
<td>1034</td>
<td>1760</td>
</tr>
<tr>
<td>30</td>
<td>( 3.28 \times 10^4 )</td>
<td>( 3.06 \times 10^4 )</td>
</tr>
<tr>
<td>40</td>
<td>( 1.05 \times 10^6 )</td>
<td>( 3.09 \times 10^5 )</td>
</tr>
<tr>
<td>80</td>
<td>( 1.10 \times 10^{12} )</td>
<td>( 2.11 \times 10^8 )</td>
</tr>
<tr>
<td>160</td>
<td>( 1.21 \times 10^{24} )</td>
<td>( 5.57 \times 10^{11} )</td>
</tr>
<tr>
<td>320</td>
<td>( 1.46 \times 10^{48} )</td>
<td>( 5.69 \times 10^{15} )</td>
</tr>
</tbody>
</table>

4. A lower bound for \( t(l) \). As before, let \( t = t(l) \) be the maximum number of \( l \)-tuples of 0's, 1's, and *'s which are pairwise comparable and compatible. In the previous section we obtained the upper bound
\[
t(l) \leq l^{\log l/0.71} = e^{\log^2 l}.
\]

The lower bound which we found is
\[
t(l) \geq 15l + 1 \quad \text{for all } l \equiv 0 \pmod{7}.
\]
The basic construction here is for \( l = 7 \), with \( t(l) = 16 \).
It is no loss of generality to assume that the array $A$ which achieves $t(l)$ rows with $l$ columns includes an all-0's row, $\vec{0}$, and an all-1's row, $\vec{1}$. Let $R$ denote the reduced $(t(l) - 2) \times l$ array when $\vec{0}$ and $\vec{1}$ and removed from $A$. Let $Z$ be the $(t(l) - 2) \times l$ matrix of all 0's, and let $J$ be the $(t(l) - 2) \times l$ matrix of all 1's. Then for any multiplicity $m$, the following array (Table 4), which is $(mt(l) - m + 1) \times (ml)$, clearly consists of rows which are pairwise comparable and compatible. This also yields the general result

$$t(ml) \geq m(t(l) - 1) + 1,$$

for all $m \geq 1$, $l \geq 1$.

| Table 4 |
|-------------------------|-------------------------|
| $\vec{0}$  $\vec{0}$  $\vec{0}$ ... $\vec{0}$  $\vec{0}$  $\vec{0}$ | $\vec{0}$  $\vec{0}$  $\vec{0}$ ... $\vec{0}$  $\vec{0}$  $\vec{0}$ |
| $Z$  $Z$  $Z$ ... $Z$  $Z$  $R$ | $Z$  $Z$  $Z$ ... $Z$  $Z$  $J$ $J$
| $\vec{0}$  $\vec{0}$  $\vec{0}$ ... $\vec{0}$  $\vec{0}$  $\vec{1}$ | $\vec{0}$  $\vec{0}$  $\vec{0}$ ... $\vec{0}$  $\vec{0}$  $\vec{1}$
| $Z$  $Z$  $Z$ ... $Z$  $R$  $J$ | $Z$  $Z$  $Z$ ... $R$  $J$  $J$
| $\vec{0}$  $\vec{0}$  $\vec{0}$ ... $\vec{0}$  $\vec{1}$  $\vec{1}$ $\vec{1}$ | $\vec{0}$  $\vec{0}$  $\vec{0}$ ... $\vec{1}$  $\vec{1}$  $\vec{1}$
| $Z$  $Z$  $Z$ ... $R$  $J$  $J$ | $Z$  $R$  $J$ ... $J$  $J$  $J$
| $\vec{0}$  $\vec{1}$  $\vec{1}$ ... $\vec{1}$  $\vec{1}$  $\vec{1}$ | $\vec{0}$  $\vec{1}$  $\vec{1}$ ... $\vec{1}$  $\vec{1}$  $\vec{1}$
| $Z$  $Z$  $R$ ... $J$  $J$  $J$ | $R$  $J$  $J$ ... $J$  $J$  $J$
| $\vec{0}$  $\vec{1}$  $\vec{1}$ ... $\vec{1}$  $\vec{1}$  $\vec{1}$ | $\vec{1}$  $\vec{1}$  $\vec{1}$ ... $\vec{1}$  $\vec{1}$  $\vec{1}$
| $\vec{1}$  $\vec{1}$  $\vec{1}$ ... $\vec{1}$  $\vec{1}$  $\vec{1}$ | $\vec{1}$  $\vec{1}$  $\vec{1}$ ... $\vec{1}$  $\vec{1}$  $\vec{1}$
\[ n = 1, \ n^2 + n + 1 = 3, \ t(3) = 5 \]

\[
\begin{array}{c}
1 \\
+ \\
0
\end{array}
\]

\[ n = 2, \ n^2 + n + 1 = 7, \ t(7) = 16 \]

\[
\begin{array}{ccc}
1 & 0 & 0 \\
+ & 1 & 1 \\
0 & * & *
\end{array}
\]

\[ n = 3, \ n^2 + n + 1 = 13, \ t(13) \geq 41 \]

\[
\begin{array}{cccc}
1 & 0 & 0 & * \\
+ & * & 1 & 1 \\
1 & * & 1 & 1 \\
0 & * & * & *
\end{array}
\]

\[ n = 4, \ n^2 + n + 1 = 21, \ t(21) \geq 86 \]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
+ & * & * & 1 \\
1 & * & * & 1 \\
1 & * & * & 1 \\
0 & * & * & *
\end{array}
\]

\[ n = 5, \ n^2 + n + 1 = 31, \ t(31) \geq 157 \]

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
+ & * & * & 1 \\
1 & * & * & 1 \\
1 & * & * & 1 \\
1 & * & * & 1 \\
0 & * & * & *
\end{array}
\]

**Table 5**

The construction by Collins, Shor and Stembridge to show that \( t(n^2 + n + 1) \geq n(n^2 + n + 1) + 2 \) for all positive integers \( n \). (Use all cyclic shifts for each of the \( n \) words of length \( l = n^2 + n + 1 \), and adjoin the vectors \( \vec{0} \) and \( \vec{1} \) consisting of \( 0 \)'s and of \( 1 \)'s respectively to obtain the dictionary.)
5. Postscript. The results presented thus far were all obtained in time for inclusion in B. Tang’s Ph.D. thesis in May, 1983. Several subsequent results on \{0, 1, *\}-sequences are presented in [7], and include the following:

i) A simpler proof of the upper bound formula,

\[ t(l) < t^{\log l}, \]

attributed to C. L. M. van Pul;

ii) The constructions illustrating \( t(1) = 2, t(3) = 5, \) and \( t(7) = 16 \) have been generalized. Three students at Eindhoven (F. Abels, W. Janse, and J. Verbakel) found three words of length 13, all of whose cyclic shifts can be used simultaneously in a dictionary, along with the “all 0’s” and “all 1’s” words, to obtain \( t(13) \geq 41 \). Three M.I.T. students (K. Collins, P. Shor, and J. Stembridge) found a general construction which yields

\[ t(n^2 + n + 1) \geq n(n^2 + n + 1) + 2 \]

for all positive integers \( n \), from which the lower bound result

\[ t(l) > cl^{3/2} \]

clearly follows. This construction is illustrated for \( 1 \leq n \leq 5 \) in Table 5.

The large gap which still remains between the upper and lower bound formulas is a clear invitation to further research.

REFERENCES


*University of Southern California,*

*Los Angeles, California,*

*AT&T Bell Laboratories,*

*Murray Hill, New Jersey*