ARTICLES

Steiner Trees on a Checkerboard

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1. Introduction

Suppose a finite set of n points are randomly scattered about in the plane. How can they be joined by a network of straight lines with the shortest possible total length? The solution to this problem has practical applications in the construction of a variety of network systems, such as roads, power lines, pipelines, and electrical circuits.

It is easy to see that the shortest network must be a tree, that is, a connected network containing no cycle. (A cycle is a closed path that allows one to travel along a connected path from a given point to itself without retracing any line.) If no *new* points can be added to the original set of points, the shortest network connecting them is called a *minimum spanning tree*.

A minimum spanning tree is not necessarily the shortest network spanning the original set of points. In most cases a shorter network can be found if one is allowed to add more points. For example, suppose you want to join three points which form the vertices of an equilateral triangle. Two sides of the triangle make up a minimum spanning tree. This spanning tree can be shortened by more than 13 percent by adding an extra point at the center and then making connections only between the center point and each corner (see FIGURE 1). Each angle at the center is 120°.

length 2   length $\sqrt{3}$

FIGURE 1

A less obvious example is the minimum network spanning the four vertices of a square. One might suppose one extra point in the center would give the minimum network, but it does not. The shortest network requires, in fact, *two* extra points (see FIGURE 2). Again all the angles around the extra points in the network are 120°. The

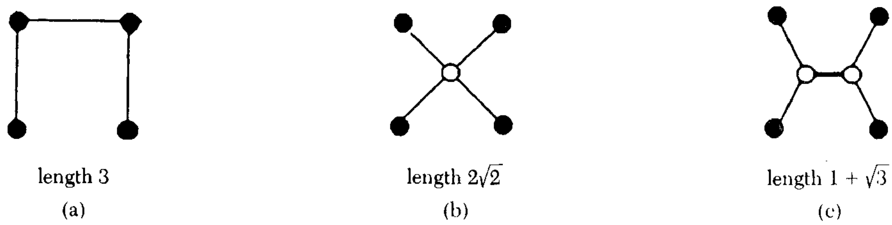


FIGURE 2

network with one extra point in the center has length $2\sqrt{2}$, or about 2.828. The network with two extra points reduces the total length to $1 + \sqrt{3}$, or about 2.732.

One of the first mathematicians to investigate such networks was Jacob Steiner, an eminent Swiss geometer who died in 1863. The extraneous points that minimize the length of the network are now called Steiner points. It has been proved that all Steiner points are junctions of three lines forming three 120° angles. The shortest network, allowing Steiner points, is called a *minimum Steiner tree.* Minimum Steiner trees are almost always shorter than minimum spanning trees, but the reduction in length usually depends on the shape of the original spanning tree. It has been conjectured [9] that for any given set of points in the plane, the length of the minimum Steiner tree cannot be less than a factor of $\sqrt{3}/2$, or about .866, times the length of the minimum spanning tree; the result has been proved, however, only for three, four, and five points [**10**], [**12**].

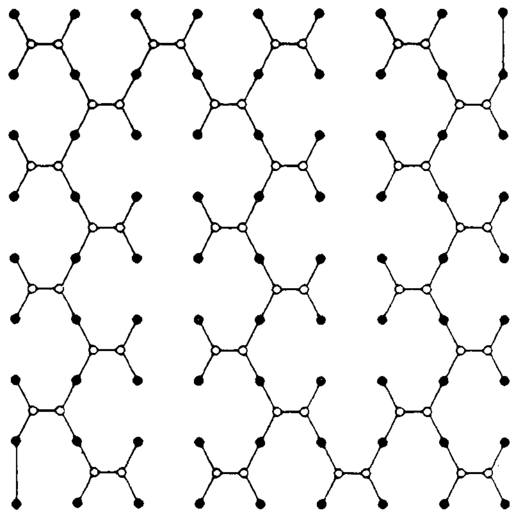
Many properties of minimum Steiner trees can be found in the excellent (but somewhat out-of-date) survey paper of E. N. Gilbert and H. O. Pollak [9]. The best current lower bound for the ratio of the minimum Steiner tree to the minimum spanning tree is .8241... (see [4]).

There are many ways to construct a minimum spanning tree. One of the simplest methods is known as a greedy algorithm, because at each step it bites off the most desirable piece. First find two points that are as close together as any other two and join them. If more than one pair of points are equally close, choose any such pair. Repeat this procedure with the remaining points in such a way that joining a pair never completes a circuit. The final result is a spanning tree of minimum length. This algorithm is due to Kruskal in a 1956 paper [11].

Given the simplicity of Kruskal's greedy algorithm for the construction of minimum spanning trees, one might suppose there would be correspondingly simple algorithms for finding minimum Steiner trees. Unfortunately, however, this is almost certainly not the case. This task belongs to a special class of "hard" problems known in computer science as NP-complete problems. When the number of points in a network is small, say 10 to 20, there are known algorithms [5], [13] for finding minimum Steiner trees in a reasonably short time. As the number of points grows, however, the computing time needed increases at a rapidly accelerating pace. Even for a relatively small number of points the best algorithms currently available could take thousands or even millions of years to terminate. Most mathematicians believe no efficient algorithms exist for constructing minimum Steiner trees on arbitrary sets of points in the plane [7], [8].

Imagine, however, that the points are arranged in a regular lattice of unit squares, like the points at the corners of the cells of a checkerboard. Is there a "good" algorithm for finding a minimum Steiner tree spanning the points of such regular patterns? In particular, what is the length of the minimum Steiner tree that joins the 81 points at the corners of a standard checkerboard? Is the tree in FIGURE 3 the solution?

Many problems involving paths through points in the plane, which are hard when



length 73.033

FIGURE 3

the points are arbitrary, become trivial when the points form regular lattices. One might expect that the task of spanning points in such arrays by minimum Steiner trees would be equally trivial. On the contrary, this problem seems to be surprisingly elusive. Up to now, only minimum Steiner trees for 2 by n rectangular arrays of points have been constructed [3]. Aside from this special case, very little seems to be known about how to find minimum Steiner trees for rectangular arrays when the number of points on each side is greater than 2.

In this paper, we will summarize various problems, conjectures, and some partial results on the minimum Steiner trees for rectangular arrays. Section 2 contains the shortest known trees for square lattices of small size. These trees consist of copies of the symmetrical tree on four points (see FIGURE 2(c)), which we call X from now on, together with a small number of "exceptional" pieces. For example, the conjectured solution for 64 points is a union of 21 X's (see FIGURE 5). In Section 3, we give a proof that a rectangular array can be spanned by a Steiner tree made up entirely of X's if and only if the array is a square and the order of the square is a power of 2. Further questions are proposed in Section 4.

2. Short Steiner trees on square lattices

In this section, we will first show the shortest Steiner trees we currently know for square lattices of size up to 14 by 14. We will then discuss a scheme for constructing Steiner trees for large square lattices from the small ones. Among all the constructions, only the patterns for the 2×2, 3×3 and 4×4 squares have been proved to be minimum Steiner trees (unpublished results of E. J. Cockayne). The constructions for square lattices of orders 2 to 9 were contained in the June 1986 issue of *Scientific American* [6]. The trees for square lattices of sizes 10 by 10 and 22 by 22 in the same article were soon improved by many readers. The current best tree for the 10 by 10 square lattice is due to one of the authors (RLG) and the best tree for the 22 by 22 square lattice is due to Eric Carlson [1]. His construction has the same total length as our general construction. Overall, the constructions fall naturally into six classes, depending on what n is modulo 6, with the rare (and remarkable) exceptions which occur when n is a power of 2. It seems that the Steiner trees for square lattices are always formed by attaching small minimum Steiner trees such as an edge E (for the 1 by 2 array), X (for the 2 by 2 array), a triangle T (formed from three vertices of a square) and L (for the 2 by 5 array). We will use the following notation:

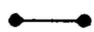
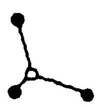
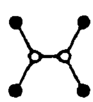
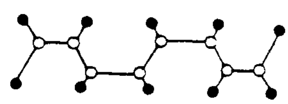
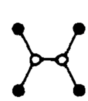
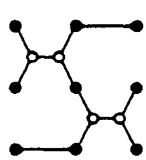
Tree	Symbol	Length
	E	$e = 1$
	T	$t = \frac{1+\sqrt{3}}{\sqrt{2}} = 1.93185\ldots$
	X	$x = 1 + \sqrt{3} = 2.73205\ldots$
	L	$l = \sqrt{35 + 20\sqrt{3}} = 8.34512\ldots$

FIGURE 4

The tree L is a minimum Steiner tree on the regular 2 by 5 array [3]. The minimum Steiner tree for the 2 by n array with n even is just made up of X's joined together by edges. On the other hand the minimum Steiner tree for the 2 by n array with n odd has length $\frac{1}{2}((n(2+\sqrt{3})-2)^2+1)^{1/2}$. The constructions for $n \times n$, $n \leqslant 14$, are illustrated in FIGURE 5.

n	Conjectured Minimum Steiner Tree	Length
2	 *	$x = 2.73205\ldots$
3	 *	$2x + 2 = 7.46410\ldots$

*known to be optimal

FIGURE 5

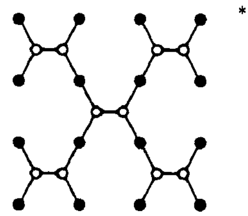
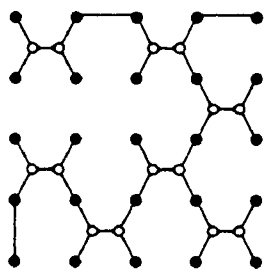
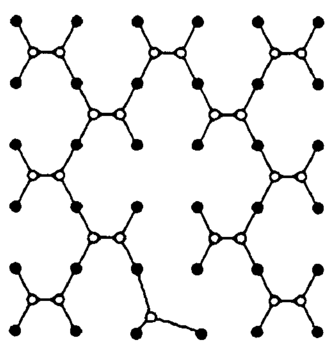
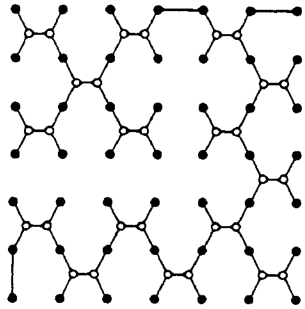
n	Conjectured Minimum Steiner Tree	Length
4		$5x = 13.66025\ldots$
5		$7x + 3 = 22.12436\ldots$
6		$11x + t = 31.98441\ldots$
7		$15x + 3 = 43.98076\ldots$

FIGURE 5 (con't)

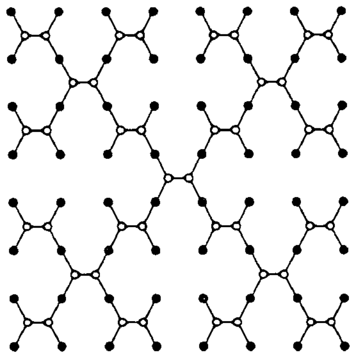
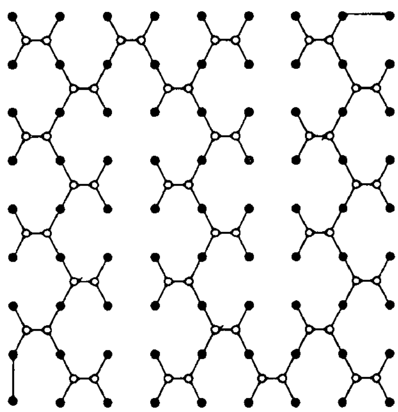
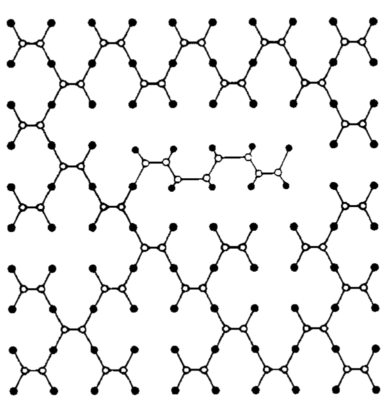
n	Conjectured Minimum Steiner Tree	Length
8		$21x = 57.373067\ldots$
9		$26x + 2 = 73.03332\ldots$
10		$30x + l = 90.30664\ldots$

FIGURE 5 (con't)

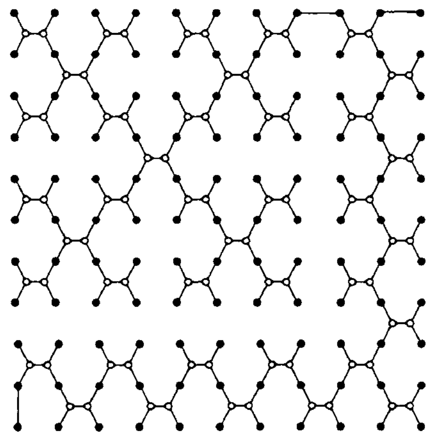
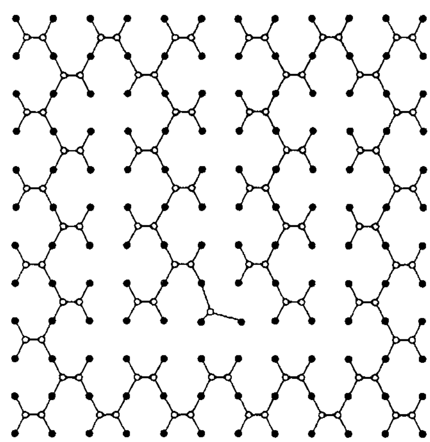
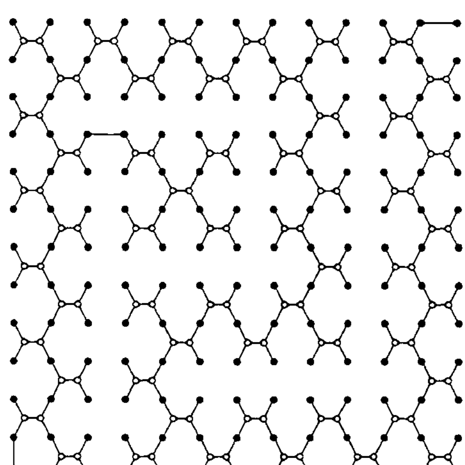
n	Conjectured Minimum Steiner Tree	Length
11		$39x + 3 = 109.54998$
12		$47x + t = 130.33824\ldots$
13		$55x + 3 = 153.26279$

FIGURE 5 (con't)

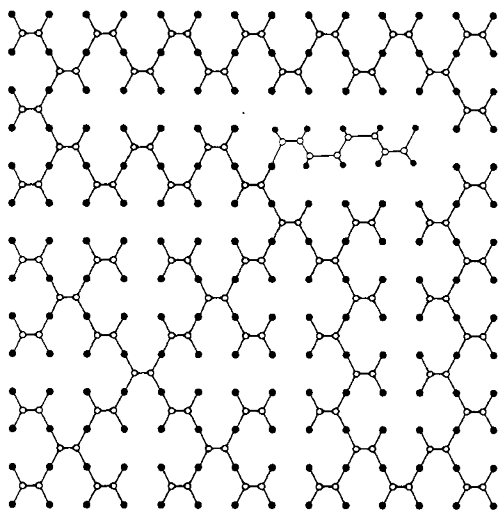
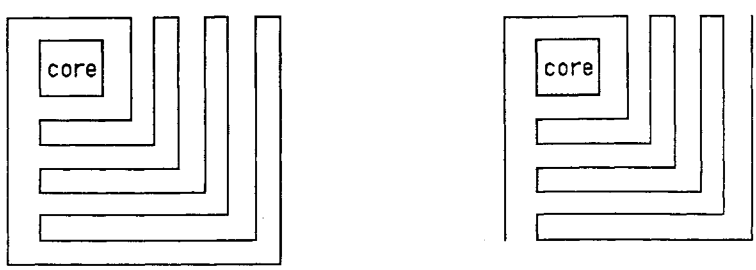
n	Conjectured Minimum Steiner Tree	Length
14		$62x + l = 177.73227$

FIGURE 5 (con't)

To construct Steiner trees for large square lattices with orders not equal to a power of 2, we will always use a "core" square with a "folded band of width 3" wrapped around it in various ways. (The only core squares we need are 6×6, 10×10, 14×14 for n even, and 0×0, 4×4 and 8×8 for n odd.) The general pattern looks like this:



n even $\qquad$ n odd

FIGURE 6

Each additional "fold" of the strip adds 6 to the size of the grid.

For example, for $n = 22$, we see that $22 \equiv 10 \pmod 6$ so we use a 10×10 core with 3 (doubled) folds of the band as shown in the picture.

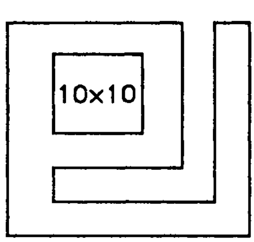
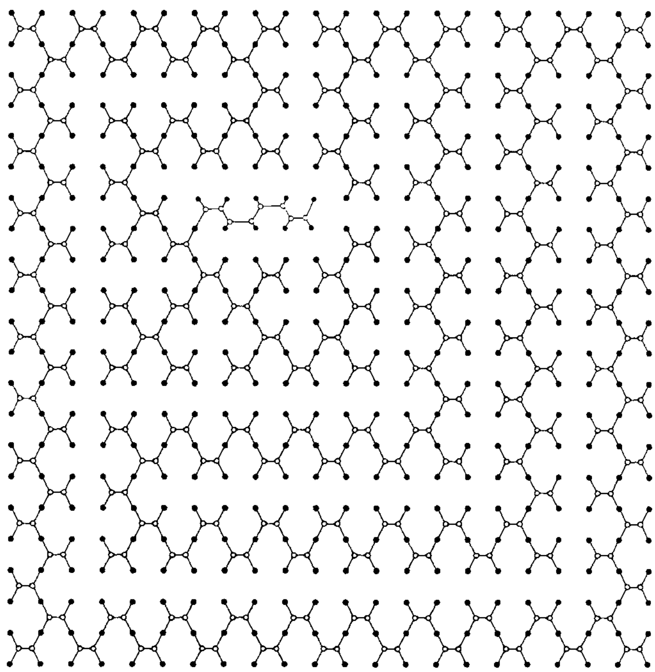


FIGURE 7

Of course, the strip must be broken and connected to the core. (See the detailed picture in FIGURE 8. We note that in order to make the connection to the strip, the corresponding place in the core must be "striplike".) When n is $6k$, the core is 0×0, i.e., empty, so we don't have to connect it to the band. In this case, we only have to break the band and reconnect the two isolated points with a T.



· Conjectured minimum Steiner tree for 22×22 array.

FIGURE 8

For *odd* n, the band doesn't form a cycle but is open at each end, leaving two isolated points at the end. You can see this happening in the conjectured minimum Steiner trees for 9×9 and 13×13. In this case, when $n \neq 0 \pmod 6$ we only need an E to connect the core to the band. Summarizing these results we have: for $n \geqslant 15$, $n \neq 2^t$:

n	Length of conjectured minimum Steiner tree for G_n
$6k$	$(12k^2 - x + t)$
$6k+1$	$(12k^2 + 4k - 1)x + 3$
$6k+2$	$(12k^2 + 8k - 2)x + l$
$6k+3$	$(12k^2 + 12k + 2)x + 2$
$6k+4$	$(12k^2 + 16k + 2)x + l$
$6k+5$	$(12k^2 + 20k + 7)x + 3$

where $x = 1 + \sqrt{3}$, $l = \sqrt{35 + 20\sqrt{3}}$, and $t = (1 + \sqrt{3})/\sqrt{2}$. Of course, for $n = 2^t$, any right-thinking person would guess that the length of the minimum Steiner tree for G_{2^t} is just $(\frac{1}{3})(4^t - 1)x$ but unfortunately we can't even prove this for $t = 3$!

3. Square lattices for powers of 2

Here we will give the proof of the main result in this paper.

THEOREM. *If a rectangular array can be spanned by a Steiner tree made up entirely of X's, then the array is a square of size 2^t by 2^t for some $t \geqslant 1$.*

Proof. We start by giving each 2×2 "cell" a pair of coordinates (i, j) in the obvious way, where the lower left-hand cell has coordinates $(0, 0)$.

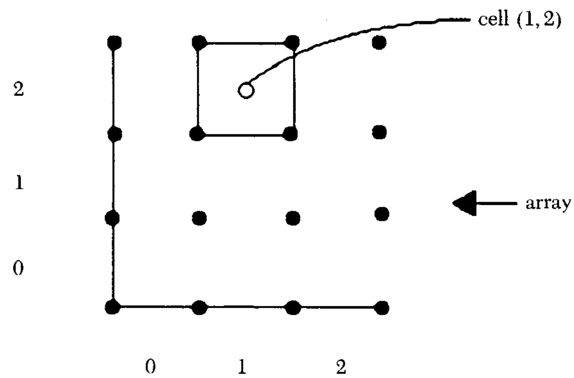


FIGURE 9

Let's call a cell (i, j) *even* if $i + j$ is even. Also, let's call a cell *occupied* if it has an X in it. *Suppose* our $a \times b$ array has an X-tree, that is a Steiner tree formed by X's.

Fact 1. Only *even* cells can be occupied.

Proof. The occupied cells must be connected. Furthermore, there can't be two *adjacent* occupied cells. Thus, occupied cells can only touch each other diagonally (as shown), in which case they are both even or both odd. However, the corner cell $(0, 0)$ is occupied and even. Thus, *all* occupied cells are even.

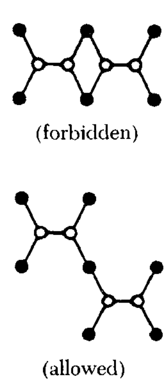


FIGURE 10

As a consequence, we see that a and b must both be even. Call an even cell (i, j) *doubly even* if i and j are both even. Otherwise, call it *doubly odd*.

Fact 2. Every doubly even cell must be occupied.

Proof. First note that *all* the even cells on the boundary must be occupied and are doubly even. Suppose we had some interior doubly even cell $(2i, 2j)$ which was not occupied. Then, we must be able to draw a path P from the center of $(2i, 2j)$ which goes to the outside of the array and doesn't pass through any occupied cell. (This is because the complement of our X-tree must be a connected set.) But if any even cell is *unoccupied* then *all four* of its even neighbors must be *occupied* (since, otherwise, one of its corner points would be disconnected.) This now implies that the only even cells P can pass through are doubly even ones, and never a doubly odd one.

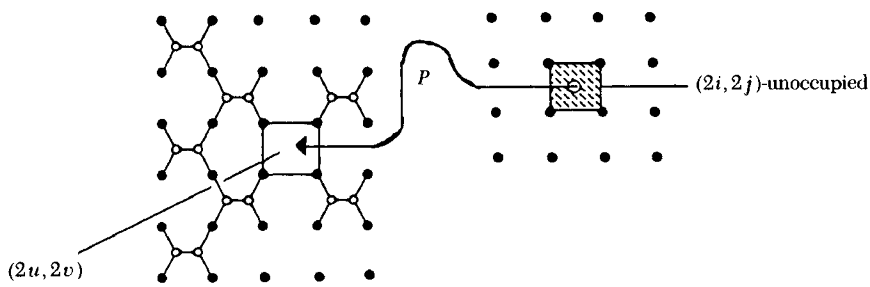


FIGURE 11

Now, focus on a doubly even cell $(2u, 2v)$ just inside the boundary which P tries to pass through on its way to the outside. Since $(2u, 2v)$ must be unoccupied (because P is going through it), all of its neighbors must be occupied. In particular, this forms a *barrier* with the adjacent occupied boundary cells which prevents P from going through to the outside here. But this happens wherever P tries to reach the outside since all the even cells on the boundary are occupied. Thus, P can never reach the outside, which is a contradiction. Hence, the hypothesis that there is an unoccupied doubly even cell is untenable, and the assertion is proved.

Therefore, in order to know what our X-tree is, we only have to know which (additional) *doubly odd* cells are occupied. Look at the picture for an 8×8 array in FIGURE 12.

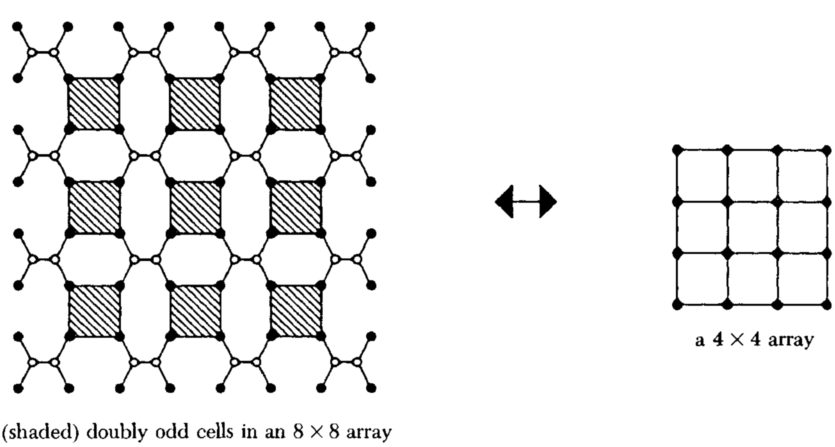


FIGURE 12

Next to the 8×8 array, a 4×4 array is drawn. Notice that there is natural correspondence between the array of 9 doubly odd cells (shaded) in the 8×8 and the array of 9 cells in the 4×4. The key observation now, which is not hard to check, is that *the set of occupied doubly odd cells in the* 8×8 *must correspond exactly to an X-tree* in the 4×4 array. Namely, consecutive doubly odd cells in the same row or column cannot both be occupied (or we get a cycle), and all points in the array must be joined together.

In effect, the doubly odd cells on the 8×8 array form a "stretched-out" version of all the cells of a 4×4 array.

More generally, this argument shows that if an $a \times b$ array has an X-tree then $a = 2A$, $b = 2B$, and, furthermore, the smaller $A \times B$ array must also have an X-tree.

We now can apply this repeatedly (similar to Fermat's method of infinite descent, except that we stop at 2×2) to get the conclusion that only $2^t \times 2^t$ arrays can have X-trees. This completes the proof of the Theorem.

Concluding Remarks

Of course, the main open problem is to determine the minimum Steiner trees for all (or even infinitely many) square lattices. It is embarrassing that even for 2^t by 2^t arrays, we still can't prove optimality for the "obviously" correct X-tree.

If we assume the distance between adjacent vertices in the lattice is 2, then it is not hard to show that any Steiner tree (minimum or not) has length of the form $\sqrt{a + b\sqrt{3}}$ where a and b are integers (possibly negative). Conceivably, this fact could be of help in proving optimality in some cases.

An interesting related question is to use only the 2×2 minimum Steiner trees and the smallest possible number of single edges (called E's) to form a spanning tree of $G(m, n)$. We know that for m and n both large enough, only a bounded number of E's are ever needed. When m is small however (where we can assume $m \leqslant n$), we may need arbitrarily many E's. Some examples of this are illustrated in FIGURE 13.

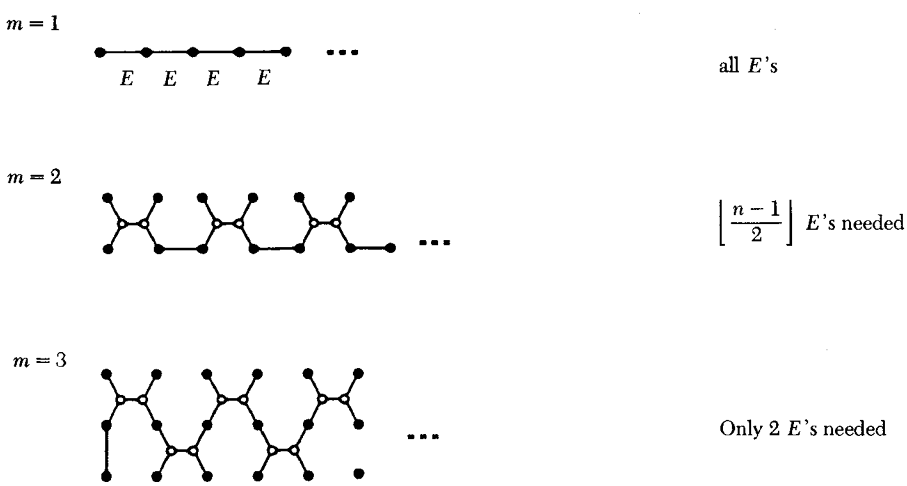


FIGURE 13

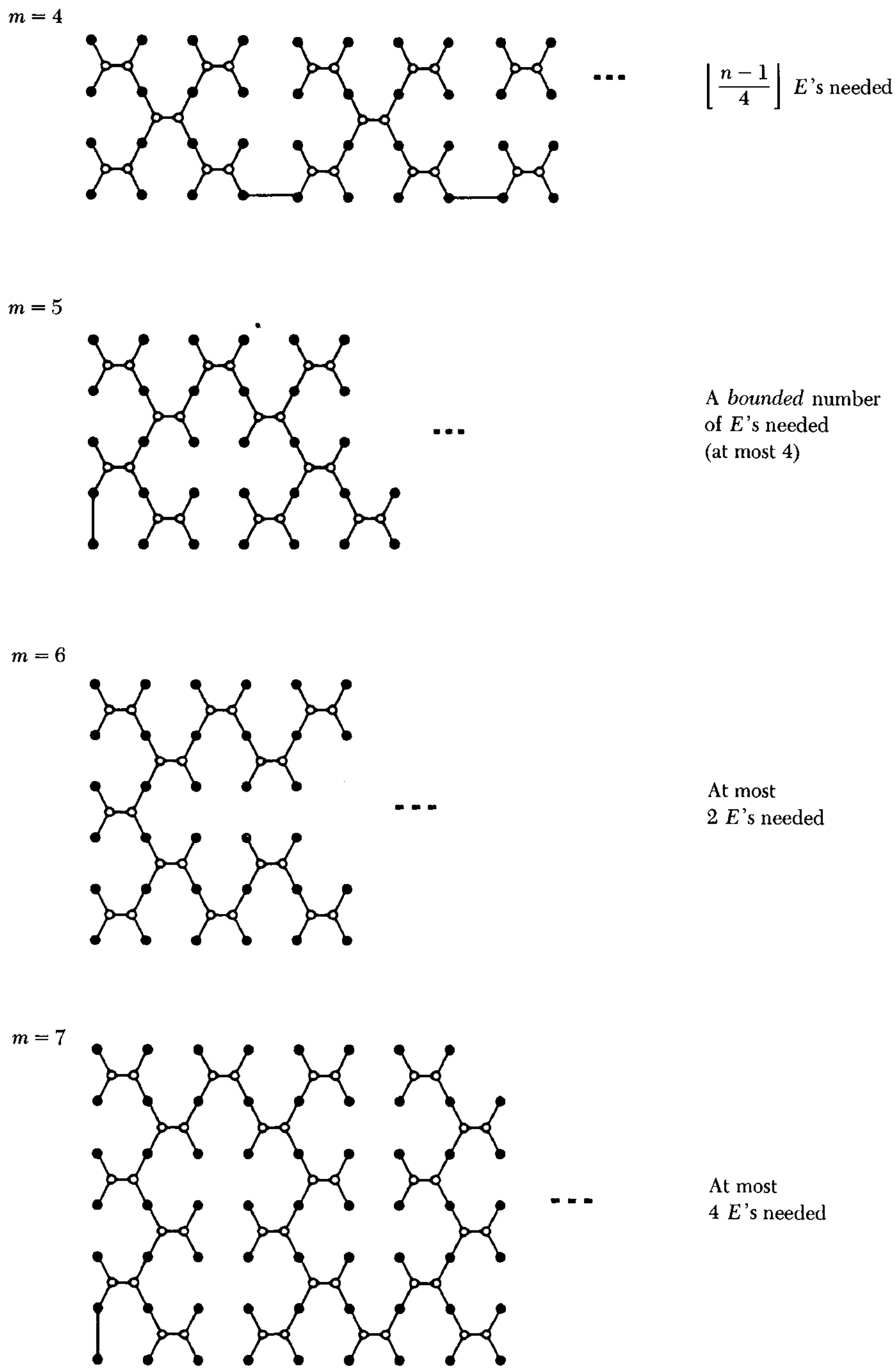


FIGURE 13 (con't)

The authors would like to acknowledge the meticulous help of Nancy Davidson in preparing the figures for this paper.

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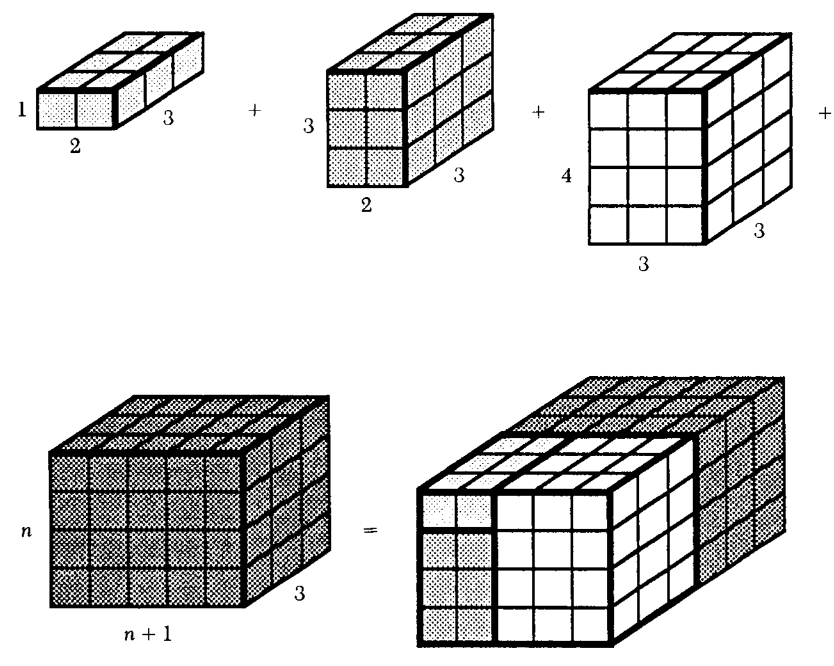
Proof without Words: Sum of Special Products

$$3(1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1))=n(n+1)(n+2).$$



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