Old and New Proofs of the Erdős-Ko-Rado Theorem

P. Frankl* and R. L. Graham

(\text{AT \& T Bell Laboratories, Murray Hill, New Jersey 07974})

ABSTRACT

The Erdős-Ko-Rado Theorem is a central result of combinatorics which opened the way for the rapid development of extremal set theory. Proofs of it are reviewed and a new generalization is given. For a survey of results related to the Erdős-Ko-Rado Theorem see [DF].

Key Words Erdős-Ko-Rado theorem, intersecting family, extremal set theory, combinatorics.

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1 Introduction

Let $X$ be a finite set of $n$ elements. Usually we suppose that $X = \{1, 2\ldots, n\}$. Let $2^X$ be the power set of $X$ and $\binom{X}{k}$ the set of all $k$-subsets of $X$. A family $\mathcal{F} \subset 2^X$ is called intersecting if $F \cap F' \neq \emptyset$ holds for all $F, F' \in \mathcal{F}$.

**Theorem 0** If $\mathcal{F} \subset 2^X$ is intersecting then

$$|\mathcal{F}| \leq 2^{n-1} \text{ holds.} \hspace{1cm} (1)$$

**Proof** There are $2^{n-1}$ pairs $\{C, X - C\}$ of complementary subsets of $X$. Since $C \cap (X - C) = \emptyset$, $|\mathcal{F} \cap \{C, X - C\}| \leq 1$ holds for each of them.

Erdős, Ko and Račo [EKR] were the first to observe the validity of (1) and they proved that there are very many families $\mathcal{F}$, achieving equality in (1).

More exactly, they proved that for every intersecting family $\mathcal{G} \subset 2^X$ there exists another intersecting family $\mathcal{F}$, $\mathcal{G} \subset \mathcal{F} \subset 2^X$, such that $|\mathcal{F}| = 2^{n-1}$ holds.

The Erdős-Ko-Rado Theorem deals with the much more difficult case when $|F| = k$ is assumed for all $F \in \mathcal{F}$, i.e., $\mathcal{F}$ is a $k$-graph.

**Theorem 1** (Erdős-Ko-Rado Theorem, special case). Suppose that $\mathcal{F} \subset$

\*Author’s Permanent affiliation: CNRS, University of Paris VII.
is intersecting, \( n \geq 2k \). Then

\[
|\mathcal{F}| \leq \binom{n-1}{k-1}.
\]  

(2)

The main purpose of the present paper is to review all known (to the authors) proofs and give some generalizations to other hypergraphs.

For an integer \( t \geq 1 \), a family \( \mathcal{F} \) is called \( t \)-intersecting if \( |F \cap F'| \geq t \) holds for all \( F, F' \in \mathcal{F} \).

To close this section let us state the general case of the Erdős-Ko-Rado Theorem.

Theorem 2 (Erdős-Ko-Rado Theorem, general case). Suppose that \( \mathcal{F} \subset \binom{X}{k} \) is \( t \)-intersecting and \( n \geq \eta_0(k, t) \). Then

\[
|\mathcal{F}| \leq \binom{n-t}{k-t}.
\]  

(3)

Remark By now it is known that the best possible value of \( \eta_0(k, t) \) is \( (k-t+1)(t+1) \) (cf., [F1] and [W]).

2 Shifting

That is how the original proof went. Since then shifting has become one of the most powerful tools in extremal set theory.

Definition 2.1 The \((i, j)\)-shift. For a family \( \mathcal{F} \subset 2^X \) and \( 1 \leq i < j \leq n \), define

\( S_{ij}(\mathcal{F}) = \{ S_{ij}(F) : F \in \mathcal{F} \} \) where

\[
S_{ij}(F) = \begin{cases} 
F' = (F - \{j\}) \cup \{i\} & \text{if } i \in F, \ i \notin F \text{ and } F' \in \mathcal{F}, \\
F & \text{otherwise}.
\end{cases}
\]

Proposition 2.2 (i) \( |S_{ij}(F)| = |F| \); (ii) \( |S_{ij}(\mathcal{F})| = |\mathcal{F}| \); (iii) If \( \mathcal{F} \) is intersecting then so is \( S_{ij}(\mathcal{F}) \).

Proof (i) and (ii) are immediate from the definition. To prove (iii), suppose by contradiction that there exist sets \( F, G \) in the intersecting family \( \mathcal{F} \) such that

\[
S_{ij}(F) \cap S_{ij}(G) = \emptyset \text{ holds.} 
\]  

(4)

Since \( F \cap G = \emptyset \) by assumption, and the only element which can be deleted is \( j \), it follows that \( F \cap G = \{i\} \).

If both \( F \) and \( G \) changed by the \((i, j)\)-shift, then \( i \in S_{ij}(F) \cap S_{ij}(G) \) would hold, contradicting (4). Thus we may assume that \( S_{ij}(F) = F, S_{ij}(G) = (G - \{j\}) \cup \{i\} \).

Similarly, \( i \in F \) would contradict (4). Thus the only reason not to replace \( F \) by \( F' = (F - \{j\}) \cup \{i\} \) during the \( i-j \)-shift is because \( F' \notin \mathcal{F} \). However, \( F' \cap G' = F \cap S_{ij}(G) = S_{ij}(F) \cap S_{ij}(F) = \emptyset \), a contradiction. ■
We can now prove (2).

**The first proof of the Erdős-Ko-Rado Theorem.** Apply induction on \( n \) and prove it simultaneously for all \( k \leq n/2 \).

(a) \( n = 2k \). We argue as with the proof of (1). The \( \binom{2k}{k} \) subsets of \( X \) can be partitioned into \( \frac{1}{2} \binom{2k}{k} = \binom{2k-1}{k-1} \) pairs of complementary sets, not both of which can be in an intersecting family. This yields \( |\mathcal{F}| \leq \binom{2k-1}{k-1} \), as desired.

(b) \( n > 2k \). Define \( \mathcal{F}_0 = \mathcal{F} \), \( \mathcal{F}_i = S_n(\mathcal{F}_{i-1}) \), \( i = 1, \ldots, n-1 \). By Proposition 2.2 we have \( |\mathcal{F}| = |\mathcal{F}_{n-1}| \), and \( \mathcal{F}_{n-1} \subseteq \binom{X}{k} \) is intersecting.

Define \( \mathcal{G} = \{ F \in \mathcal{F}_{n-1} : n \notin F \} \), \( \mathcal{H} = \{ F - \{ n \} : n \in F \in \mathcal{F} \} \).

Clearly \( |\mathcal{F}| = |\mathcal{G}| + |\mathcal{H}| \), \( \mathcal{G} \subseteq \mathcal{H} \), and thus by induction \( |\mathcal{G}| \leq \binom{n-2}{k-1} \) holds. Consequently, \( |\mathcal{H}| \leq \binom{n-2}{k-2} + \binom{n-2}{k-2} = \binom{n-1}{k-1} \) would be sufficient to show \( |\mathcal{F}| \leq \binom{n-2}{k-1} \).

The desired upper bound for the cardinality of \( \mathcal{H} \subseteq \binom{\{1, 2, \ldots, n-1\}}{k-1} \) will follow from the induction hypothesis once we prove the following.

**Proposition 2.3** \( \mathcal{H} \) is intersecting.

**Proof** Suppose the contrary, i.e., there exist disjoint sets \( H, H' \in \mathcal{H} \). Since \( |H \cup H'| = 2(k-1) < n-1 \), there exists some \( i, 1 \leq i < n \) satisfying \( i \notin H \cup H' \). By definition \( F = H \cup \{ n \} \) is in \( \mathcal{F}_{n-1} \). Since \( n \in F \) then \( F \in \mathcal{F} \), and consequently, \( F \in \mathcal{F}_i \) holds for all \( 1 \leq i \leq n-1 \). This means that \( S_n(F) = F \), i.e., \( F \) did not get replaced during the \((i, n)\)-shift. This can happen only if \( (F - \{ n \}) \cup \{ i \} = (H \cup \{ i \}) \in \mathcal{F}_{i-1} \) and consequently \( (H \cup \{ i \}) \in \mathcal{F}_{n-1} \) hold.

However, \( (H \cup \{ i \}) \cap (H' \cup \{ n \}) = \emptyset \), a contradiction. \( \square \)

### 3 Shadows

Given a \( k \)-graph \( \mathcal{F} \) and an integer \( l, 1 \leq l \leq k \), the \( l \)-shadow \( \sigma_l(\mathcal{F}) \) is defined as follows.

\[
\sigma_l(\mathcal{F}) = \{ G : |G| = l \}, \quad \text{and for some } F \in \mathcal{F}, G \subseteq F \}.
\]

Given an integer \( m \) and a \( k \)-graph \( \mathcal{F} \) of cardinality \( m \), what can one say
about $|\sigma_i(F)|$? Clearly, $|\sigma_i(F)| \leq \binom{k}{l} |F|$ holds, with equality if and only if $|F \cap F'| < l$ holds for all distinct $F, F' \in \mathcal{F}$.

The real problem is to get best possible lower bounds. The answer is given by the Kruskal-Katona Theorem, one of the most widely used results concerning finite sets. We shall only state and prove a numerical consequence of it which is due to Lovász.

Kruskal-Katona Theorem ([Kr], [Ka2], [I1]). Let $\mathcal{F}$ be a $k$-graph, and suppose $|\mathcal{F}| \geq \binom{x}{k}$ with $x \geq k$, real. Then

$$|\sigma_l(\mathcal{F})| \geq \binom{x}{l}$$

holds for all $0 \leq l \leq k$. (5)

First note that it is sufficient to prove (5) for the case $l = k - 1$ (and then apply this case $k - l$ times noting the monotonicity of $\binom{x}{s}$ as a function of $x$).

The proof which we are going to present is from [F2] and is based upon the fact that the $(i, j)$-shift does not increase the shadow.

Proposition 3.1 Let $\mathcal{F} \subset \binom{X}{k}$ be a $k$-graph, and suppose $1 \leq i < j \leq n$. Then

$$\sigma_{i-1}(S_{ij}(\mathcal{F})) \subset S_{ij}(\sigma_{i-1}(\mathcal{F})).$$

(6)

Equation (6) can be proved by a relatively simple case by case analysis, which we leave to the reader.

Define inductively $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_i = S_{ij}(\mathcal{F}_{i-1})$, $2 \leq i \leq n$. In view of (6) we have $|\sigma_{i-1}(\mathcal{F}_n)| \leq |\sigma_{i-1}(\mathcal{F})|$. Therefore it is sufficient to deal with $\mathcal{F}_n$.

Recall the definitions: $\mathcal{F}_n(1) = \{ F \setminus \{ 1 \} : F \in \mathcal{F}_n \}$, $\mathcal{F}_n(\bar{1}) = \{ F \in \mathcal{F}_n : 1 \notin F \}$.

Claim 3.2. (i) $\sigma_{i-1}(\mathcal{F}_n) \supseteq |\mathcal{F}_n(1)| + |\sigma_{i-2}(\mathcal{F}_n(1))|$; (ii) $\sigma_{i-1}(\mathcal{F}_n(\bar{1})) \subset \mathcal{F}_n(1)$.

Proof of the claim By definition, $\mathcal{F}_n(1) \subset \sigma_{i-1}(\mathcal{F}_n)$ and $\{ 1 \cup G : G \in \sigma_{i-2}(\mathcal{F}_n(1)) \} \subset \sigma_{i-1}(\mathcal{F}_n)$ hold. Moreover, these two families are disjoint, proving (i).

To prove (ii) choose $(G, H)$ with $H \in \mathcal{F}_n(\bar{1})$, $G \subset H$, $|G| = k - 1$. Let $i$ be the unique element of $H - G$.

The only way $H$ was not replaced by $H' = G \cup \{ i \}$ when $S_{11}$ was applied is that $H' \in \mathcal{F}_{i-1}$ and, consequently, $H' \in \mathcal{F}_n$. This proves $G \in \mathcal{F}_n(1)$, as desired.

Now the proof of (6) is easy. Apply induction on $k$ and for given $k$, on $|\mathcal{F}|$; of course, the case $|\mathcal{F}| = 1$ is trivial. We distinguish two cases.

(a) $|\mathcal{F}_n(1)| \geq \binom{x - 1}{k - 1}$. By induction $|\sigma_{i-2}(\mathcal{F}_n(1))| \geq \binom{x - 1}{k - 2}$ and thus
by (i), $|\sigma_{k-1}(\mathcal{F}_n)| \geq \binom{x-1}{k-1} + \binom{x-1}{k-2} = \binom{x}{k-1}$ follow.

(b) $|\mathcal{F}_n(1)| < \binom{x-1}{k-1}$. In view of (ii), $|\mathcal{F}_n(1)| \geq k$ and thus $x-1 \geq k$ follows. On the other hand,

$$|\mathcal{F}_n(1)| = |\mathcal{F}| - |\mathcal{F}_n(1)| \geq \binom{x}{k} - \binom{x-1}{k-1} = \binom{x-1}{k}.$$ 

Applying the induction hypothesis and using (ii), $|\mathcal{F}_n(1)| \geq |\sigma_{k-1}(\mathcal{F}_n(1))| \geq \binom{x-1}{k-1}$ follows, a contradiction. ■

Erdős-Ko-Rado from Kruskal-Katona (cf. [Da], [Ka1]). Suppose that\n
$\mathcal{F} \subseteq \binom{X}{k}$, $n \geq 2k$, and $|\mathcal{F}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$. Define the complementary family\n
$\mathcal{G} = \{X - F : F \in \mathcal{F}\} \subseteq \binom{X}{n-k}$. By (5) $|\sigma_i(\mathcal{G})| \geq \binom{n-1}{k}$, and thus $|\mathcal{F}| + |\sigma_i(\mathcal{G})| > \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ holds.

Consequently there is some $F \in \mathcal{F} \cap \sigma_i(\mathcal{G})$. Since $F \in \sigma_i(\mathcal{G})$, $F \subseteq X - F'$ for some $F' \in \mathcal{F}$. This implies $F \cap F' = \emptyset$, i.e., $\mathcal{F}$ is not intersecting. ■

4 Cyclic Permutations

In this section we reproduce the short proof of (2) given by Katona [Ka3].

For any of the $(n-1)!$ cyclic permutations $\pi : a_1, \ldots, a_n$ of 1, 2, \ldots, $n$, consider the $k$-graph $\mathcal{F}(\pi)$ consisting of the $n$ blocks of length $k$ (for each $i$ there is one such block starting at $a_i$ namely, $a_i, a_{i+1}, \ldots a_{i+k-1}$ (index addition is performed modulo $n$).

Claim 4.1 Suppose that $\mathcal{G} \subseteq \mathcal{F}(\pi)$ is an intersecting subfamily. Then $|\mathcal{G}| \leq k$.

Proof Without loss of generality, we may assume that $A = \{a_1, \ldots, a_i\} \subseteq \mathcal{G}$.

Being intersecting implies that for each $G \subseteq \mathcal{G}$ either its first or last element is in $A$. A priori this gives $2(k-1)$ more candidates outside $A$ for membership in $\mathcal{G}$.

We can group them into $k-1$ pairs by associating the one ending in $a_i$ with the one starting in $a_{i+1}$, $1 \leq i \leq k-1$. Since the two $k$-sets are disjoint in each pair, $|\mathcal{G}| \leq 1 + k - 1 = k$ follows. ■

Let us now estimate the number $M$ of pairs $(F, \pi)$, where $F \in \mathcal{G}$, $\pi$ is a cyclic permutation and $F \in \mathcal{F}(\pi)$.

For each $k$-set $F$ there are $k!(n-k)!$ cyclic permutations $\pi$ with $F \in \mathcal{F}(\pi)$,
Thus \(M = |\mathcal{F}|k!(n-k)!\).

On the other hand, claim 4.1 implies \(M \leq k(n-1)\). Consequently, \(|\mathcal{F}| \leq \frac{k!(n-1)!}{k!(n-k)!} = \binom{n-1}{k-1},\) follows as desired.

5 Delsarte’s Linear Programming Bound: Lovász Proof

Let \(A_1, \ldots, A_{\binom{n}{k}}\) be an arbitrary fixed ordering of all the \(k\)-subsets of \(X\). For an intersecting family \(\mathcal{F} \subseteq \binom{X}{k}\), let \(v(F) = (v_1, \ldots, v_{\binom{n}{k}})\) be its characteristic vector, i.e., \(v_i = 1\) or 0 according to whether \(A_i \in \mathcal{F}\) or \(A_i \notin \mathcal{F}\) holds.

Let \(B\) be an \(\binom{n}{k} \times \binom{n}{k}\) real symmetric matrix whose general entry \(b_{ij}\) satisfies \(b_{ij} = 0\) whenever \(A_i \cap A_j = \emptyset\).

Let \(I\) and \(J\) denote the \(\binom{n}{k} \times \binom{n}{k}\) identity, and all \(I\)'s matrices, respectively.

**Claim 5.1** If \(B + I - cJ\) is positive semi-definite for some positive \(c\), then \(|\mathcal{F}| \leq 1/c\).

**Proof** Consider \(y = v(B + I - cJ)v^T\). By assumption, \(vBv^T = 0\). Also \(vIv^T = |\mathcal{F}|\) and \(v\mathcal{F}v^T = |\mathcal{F}|^2\) are immediate from the definition. Thus \(y = |\mathcal{F}| - c|\mathcal{F}|^2\).

On the other hand, \(y \geq 0\) follows from positive semi-definiteness, i.e., \(c|\mathcal{F}|^2 \leq |\mathcal{F}|\), or equivalently, \(|\mathcal{F}| \leq 1/c\).

Following Lovász [12] we define \(B = (b_{ij})\) by \(b_{ij} = \begin{cases} \binom{n-k-1}{k-1}^{-1} & \text{if } A_j \cap A_i = \emptyset \\ 0 & \text{otherwise} \end{cases}\)

To prove (2) we need to compute the eigenvalues of \(B\). Obviously, the all \(I\)'s vector is a common eigenvector of \(B\), \(I\) and \(J\) with respective eigenvalues \(\frac{n-k}{k}\), \(1\) and \(\binom{n}{k}\). Thus it is annihilated by \(B = B + I - \binom{n-1}{k-1}J\).

To prove positive semi-definiteness we have to show that all the remaining eigenvalues of \(B\) are at least \(-1\). It is easy to give eigenvectors having eigenvalue \(-1\), namely for each pair \((x, y), x, y \in X\) define \(v(x, y) = (v_1, \ldots, v_{\binom{n}{k}})\) by

\[
v_i = \begin{cases} 
1 & \text{if } A_i \cap \{x, y\} = \{x\}, \\
-1 & \text{if } A_i \cap \{x, y\} = \{y\}, \\
0 & \text{otherwise}.
\end{cases}
\]

Direct computation shows that \(v(x, y)B = -v(x, y)\) holds. The \(v(x, y)\) span a vector space of dimension \(n-1\). To find the remaining \(\binom{n}{k} - (n-1) - 1\) eigenvectors
we have to do some more work. For \(2 \leq i \leq k\) and any two disjoint \(i\)-element subsets \(C = \{x_1, \ldots, x_i\}, D = \{y_1, \ldots, y_i\}\) of \(X\), define the vector \(u(C, D) = (u_1, \ldots, u_n)\) by

\[
u_i = \begin{cases} \frac{(-1)^{|D \cap A|}}{\binom{n-A}{k-i}} & \text{if } |A \cap \{x_l, y_l\}| = 1 \text{ holds for } 1 \leq l \leq i, \\ 0 & \text{otherwise}. \end{cases}
\]

**Claim 5.2**

\[
u(C, D) = (-1)^i \left( \frac{n-k-i}{k-i} \right) \left( \frac{n-k-1}{k-1} \right) u(C, D).
\]

**Proof** Set \(\delta = \left( \frac{n-k-1}{k-1} \right)^{-1}\) and compute the \(r\)th entry \(v_r\) of \(u(C, D)B\). This is simply the dot product of \(u_i\) and the column of \(B\) belonging to the set \(A_r\).

Suppose first that \(|A \cap \{x_l, y_l\}| = 1\) for \(1 \leq l \leq i\). The only way to get a non-zero (actually \(\delta\) or \(-\delta\)) is for \(A \in \binom{X}{k}\) to satisfy \(A \cap (C \cup D) = (C \cup D) - A\), i.e., if \(A\) and \(A_r\) are complementary inside \(C \cup D\). Consequently, \(v_r = (-1)^i \delta \binom{n-k-1}{k-i} u_r\), follows.

If \(|A \cap \{x_l, y_l\}| = 2\) for some \(l\), then there is no way to get a non-zero term, yielding \(v_r = 0\).

If \(A \cap \{x_l, y_l\} = \emptyset\) holds, then we can associate to every position \(A\) giving a non-zero term and satisfying \(A \cap \{x_l, y_l\} = \{x_l\}\) the position \((A - \{x_l\}) \cup \{y_l\}\) giving a non-zero term exactly of the opposite sign. This gives again \(v_r = u_r = 0\).

Next we are going to exhibit \(\binom{n}{i} - \binom{n}{i-1}\) linearly independent vectors \(u(C, D)\). This will show that the eigenspace belonging to the eigenvalue \((-1)^i \binom{n-k-i}{k-i} / \binom{n-k-1}{k-1}\) has dimension at least \(\binom{n}{i} - \binom{n}{i-1}\) for \(i = 0, 1, \ldots, k\). Since these numbers sum up to \(\binom{n}{k}\) equality holds everywhere. Consequently, we have found all the eigenvalues and the positive semi-definiteness follows from \(\binom{n-k-1}{k-1}\), valid for \(1 \leq i \leq k\).

If \(C = \{x_1, \ldots, x_i\}\) and \(D = \{y_1, \ldots, y_i\}\) with \(x_i < y_i\) then we write \(C < D\). It is not hard to see that there are exactly \(\binom{n}{i} - \binom{n}{i-1}\) sets \(D \in \binom{X}{i}\) for which some \(C \in \binom{X}{i}\) satisfying \(C < D\) exists. Fix some \(C = C(D)\) with this property for each such \(D\).

Then the vectors \(u(C(D), D)\) are linearly independent, and we are done.
6 Stronger results for large $n$

Let $L = \{l_1, \ldots, l_s\}$ with $0 \leq l_1 < \ldots < l_s < k$. A family $\mathcal{F} \subseteq \binom{X}{k}$ is called an $L$-system if $|F \cap F'| \subseteq L$ holds for all distinct $F, F' \in \mathcal{F}$.

In this terminology $\mathcal{F}$ is $t$-intersecting iff it is an $L$-system with $L = \{t, t+1, \ldots, k-1\}$.

In this section we are going to prove the following general result.

**Theorem 6.1** [DEF]. Suppose that $\mathcal{F} \subseteq \binom{X}{k}$ is an $L$-system. Then

$$|\mathcal{F}| \leq \prod_{l \in L} \frac{n-l}{k-l}$$

holds for $n \geq k \left(\frac{3k}{k}\right)$.

Note that for $L = \{t, \ldots, k-1\}$, the upper bound becomes $\left(\frac{n-t}{k-t}\right)$, so that this result generalizes the Erdős-Ko Rado Theorem.

**Proof** Apply induction on $|L| = s$. In the case $s = 0$, the upper bound $|\mathcal{F}| \leq 1$ is trivially true. Supposing the theorem is true for $s-1$, we attack the case $|L| = s$.

(a) $l_1 = 0$. For each $x \in X$ the family $\tilde{\mathcal{F}}(x) = \{F \in \mathcal{F} : x \in F\}$ is an $L'$-system with $L' = \{l_2, \ldots, l_s\}$. Thus by induction

$$|\tilde{\mathcal{F}}(x)| \leq \prod_{2 \leq i \leq s} \frac{n-l_i}{k-l_i}.$$ 

Since $\sum_{x \in X} |\tilde{\mathcal{F}}(x)| = k \cdot |\mathcal{F}|$ holds, we obtain

$$|\mathcal{F}| \leq \frac{n}{k} \prod_{2 \leq i \leq s} \frac{n-l_i}{k-l_i} = \prod_{i \in L} \frac{n-l_i}{k-l_i}$$

as desired.

(b) $l_1 > 0$. If $|F \cap F'| \neq l_1$ holds for all $F, F' \in \mathcal{F}$ then $\mathcal{F}$ is actually an $\{l_2, \ldots, l_s\}$-system and the much stronger bound $|\mathcal{F}| \leq \prod_{2 \leq i \leq s} \frac{n-l_i}{k-l_i}$ follows by induction.

Suppose next that $|F_1 \cap F_2| = l_1$ for some $F_1, F_2 \in \mathcal{F}$. Set $G = F_1 \cap F_2$. If $G \subseteq F$ for all $F \in \mathcal{F}$, then replacing $\mathcal{F}$ by $\{F - G : F \in \mathcal{F}\}$, $k$ by $k-l_1$ and $L$ by $\{0, l_2-l_1, \ldots, l_s-l_1\}$ brings us back to case (a).

Suppose finally that $G \nsubseteq F_3$ holds for some $F_3 \in \mathcal{F}$.

**Claim 6.2** $|F \cap (F_1 \cup F_2 \cup F_3)| > l_1$ holds for all $F \in \mathcal{F}$.

**Proof** Since $\mathcal{F}$ is an $L$-system, $|F \cap F_i| \geq l_1$ for $i = 1, 2, 3$. If $|F \cap (F_1 \cup$
For each $H \in \binom{F_1 \cup F_2 \cup F_3}{l_1 + 1}$, define $\mathcal{F}(H) = \{F \in \mathcal{F} : H \subseteq F\}$. Then $\mathcal{F}(H)$ is an $\{l_2, \ldots, l_s\}$-system. Thus the induction hypothesis and the claim imply:

$$|\mathcal{F}| \leq \sum_{n \leq\binom{F_1 \cup F_2 \cup F_3}{l_1 + 1}} \prod_{2 \leq i \leq s} \binom{n - l_i}{k - l_i} \leq \binom{3k}{k},$$

and the statement of the theorem follows provided $\frac{n - l_1}{k - l_1} > \binom{3k}{k}$. $\blacksquare$

Remark From the proof it is clear that for $l_1 > 0$ equality can hold only if all $F \in \mathcal{F}$ contain a fixed $l_1$-element set.

### 7 Edge-fillings by pairs

Let $\mathcal{F}$ be an intersecting $k$-graph. we call $\mathcal{F}$ non-trivial if $\bigcap_{i \in I} F_i = \emptyset$ holds.

Let us call a family $\mathcal{G}$ an edge-filling of $\mathcal{F}$ if for every $F \in \mathcal{F}$ there exists some $G \in \mathcal{G}$ satisfying $G \subseteq F$.

**Theorem 7.1** Every non-trivial intersecting family $\mathcal{F} \subseteq \binom{X}{k}$ possesses an edge-filling $\mathcal{G} \subseteq \binom{X}{2}$ satisfying $|\mathcal{G}| \leq k^2 - k + 1$.

**Proof** If $\mathcal{F}$ is 2-intersecting, then $\binom{F}{2}$ is an edge-filling of $\mathcal{F}$ for every $F \in \mathcal{F}$. Thus we may assume that there exist $F_1, F_2 \in \mathcal{F}$ with $F_1 \cap F_2 = \{x\}$ for some $x \in X$.

Since $\mathcal{F}$ is non-trivial, $x \notin F_3 \in \mathcal{F}$ holds for some $F_3 \in \mathcal{F}$.

Now the theorem follows from the following claim.

**Claim 7.2** $\mathcal{G} = \{(y, z) : y \in F_1 - \{x\}, z \in F_2 - \{x\}\} \cup \{\{x, u\} : u \in F_3\}$ is an edge-filling of $\mathcal{F}$.

**Proof** Take an arbitrary $F \in \mathcal{F}$ and distinguish two cases.

(a) $x \notin F$. Since $\mathcal{F}$ is intersecting, $F \cap (F_i - \{x\}) = \emptyset$ holds for $i = 1, 2$. Consequently $F$ contains one of the 2-subsets in the first part of $\mathcal{G}$.

(b) $x \in F$. Again, since $\mathcal{F}$ is intersecting, $F \cap F_3 = \emptyset$, so that $F$ must contain one of the 2-subsets in the second part of $\mathcal{G}$. $\blacksquare$ $F$ has to contain one of the 2-subsets in the second part of $\mathcal{G}$.

Remark More careful analysis shows that one can find an edge-filling $\mathcal{G}$ with $|\mathcal{G}| \leq k^2 - k + 1$ unless $\mathcal{F}$ consists of the $k^2 - k + 1$ lines of a projective plane.
of order \( k - 1 \). We hope to return to this and more general problems in a subsequent paper.

### 8 The Erdős-Ko-Rado Theorem for General Hypergraphs

Let \( \mathcal{H} \) be a \( k \)-graph. For a vertex \( x \) of \( \mathcal{H} \) its degree is the number of edges of \( \mathcal{H} \) which contain \( x \). Clearly, these edges form an intersecting hypergraph. Let \( \Delta(\mathcal{H}) \) denote the maximum degree of \( \mathcal{H} \).

We say that \( \mathcal{H} \) has the Erdős-Ko-Rado property if \( \Delta(\mathcal{H}) \geq |\mathcal{F}| \) holds for all intersecting subfamilies \( \mathcal{F} \subset \mathcal{H} \).

For a set \( I \), recall the definition \( \mathcal{H}(I) = \{ H - I : I \subseteq H \in \mathcal{H} \} \). Let \( \Delta_i(\mathcal{H}) \) denote the maximum of \( |\mathcal{H}(I)| \) over all \( i \)-sets \( I \). Clearly, \( \Delta_i(\mathcal{H}') = \Delta_i(\mathcal{H}) \) and \( \Delta_{k-1}(\mathcal{H}) = 1 \).

**Theorem 8.1** Suppose that \((k^2 - k + 1) \Delta_2(\mathcal{H}) \leq \Delta_1(\mathcal{H})\) holds. Then \( \mathcal{H} \) has the Erdős-Ko-Rado property.

**Proof** Let \( \mathcal{F} \subset \mathcal{H} \) be intersecting. If \( \bigcap_{i \in I} F_i \neq \emptyset \), then \( |\mathcal{F}| \leq \Delta_1(\mathcal{H}) \) is immediate. Thus we may suppose that \( \mathcal{F} \) is non-trivial. By Theorem 7.1 there exists an edge-filling \( \mathcal{S} \) of \( \mathcal{F} \), consisting of 2-element sets and satisfying \( |\mathcal{S}| \leq k^2 - k + 1 \). This implies

\[ |\mathcal{F}| \leq \sum_{G \in \mathcal{S}} |\mathcal{H}(G)| \leq \sum_{G \in \mathcal{S}} |\mathcal{H}(G)| \leq (k^2 - k + 1) \Delta_2(\mathcal{H}) \leq \Delta_1(\mathcal{H}). \]

**Remark** Since \( \Delta_1(\mathcal{H}) = \binom{n-1}{k-1} \) and \( \Delta_2(\mathcal{H}) = \binom{n-2}{k-2} \) hold for \( \mathcal{H} = \binom{X}{k} \), Theorem 8.1 implies (2) for \( n > (k-1)(k^2 - k + 1) \). As we shall see later, it implies (3) as well.

Now we give a construction showing that the theorem is in a sense best possible. Let \( k - 1 \) be an integer such that there is a projective plane of order \( k - 1 \).

If \( n \) is a sufficiently large multiple of \( k^2 - k + 1 \) then one can find \( k - 1 \) orthogonal partitions of \( X \) into \((k^2 - k + 1)\)-element sets. That is, there exist \( B_{ij} \in \binom{X}{k^2 - k + 1}, 1 \leq i < k, 1 \leq j \leq n/(k^2 - k + 1) \), such that \( B_{ij} \cup \cdots \cup B_{in} = X \), \( 1 \leq i < k \), and \( |B_{ij} \cap B_{in}| \leq 1 \) for \( i \neq s \) and all \( j, t \).

Now form a \( k \)-graph \( \mathcal{H} \) by replacing each \( B_{ij} \) by the set of lines of a projective plane of order \( k - 1 \) on \( B_{ij} \). Then \( |\mathcal{H}| = (k-1)n \), \( \Delta(\mathcal{H}) = (k-1)k = k^2 - k \) and \( \Delta_2(\mathcal{H}) = 1 \). However, the size of the largest intersecting subfamily of \( \mathcal{H} \) is \( (k^2 - k + 1) \), namely, each projective plane of order \( k - 1 \) gives such an example.

Let us conclude this section with a well known open problem. Call \( \mathcal{H} \subset 2^X \) a complex, if \( G \subseteq H \subseteq \mathcal{H} \) implies \( G \subseteq \mathcal{H} \).

**Conjecture 8.2** (Chvátal [C]) Suppose that \( \mathcal{H} \) is a complex, and \( \mathcal{F} \subset \mathcal{H} \) is intersecting. Then \( |\mathcal{F}| \leq \Delta(\mathcal{H}) \) holds.
REFERENCES


厄多斯-柯-拉多定理的新老证明

P. 弗兰克尔，R. L. 格拉穆

(AT&T 贝尔实验室)

摘 要

厄多斯-柯-拉多定理是组合论的一个主要结果，它开辟了极值集论迅速发展的道路。本文回顾了它的多种证明，并给出了一个新的推广。有关该定理的综合报告见[DF]。

关键词 厄多斯-柯-拉多定理, 优雅, 极值集论, 组合论。