BOUNDS FOR ARRAYS OF DOTS WITH DISTINCT SLOPES OR LENGTHS

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An $n \times m$ sonar sequence is a subset of the $n \times m$ grid with exactly one point in each column, such that the $\binom{m}{2}$ vectors determined by them are all distinct. We show that for fixed $n$ the maximal $m$ for which a sonar sequence exists satisfies $n - Cn^{11/20} < m < n + 4n^{2/3}$ for all $n$ and $m > n + c\log n \log \log n$ for infinitely many $n$.

Another problem concerns the maximal number $D$ of points that can be selected from the $n \times m$ grid so that all the $\binom{D}{2}$ vectors have slopes. We prove $n^{1/2} \ll D \ll n^{4/5}$.

An $n \times m$ sonar sequence [3] is an array of dots and blanks having $n$ rows and exactly one dot in each of its $m$ columns, subject to the requirement that distinct pairs of dots determine distinct vectors. Any two such vectors must differ in slope or in length. Whenever $p$ is prime, an example of a $p \times p$ sonar sequence $a_1, a_2, \ldots, a_p$ is given by letting $a_i \equiv i^2 \pmod{p}$, and choosing $1 \leq a_i \leq p$, so that $a_i$ gives the row coordinate of the dot in the $i^{th}$ column.

Recalling the size of gaps between primes this example shows that in trying to maximize $m$, it is always possible to achieve $m > n - n^{11/20}$. By Theorem 4, which uses the method of [2], an upper bound for large $n$ is $m < n + 4n^{2/3}$. Theorem 5 uses a new result in [5] to say that for infinitely many $n$, there exists an $n \times m$ sonar sequence with $m > n + c\log n \log \log n$.

Now consider an $n \times n$ array having $D$ dots. Robert Peile raised the question of maximizing $D$ when the vectors determined by different pairs of dots are required to differ in slope. Theorem 1 tells us that $D \leq 5n^{4/5}$ for large $n$. Theorem 2 by algebraic construction shows that $D > (1/2 + o(1))n^{1/2}$. In contrast, random choice in Theorem 3 guarantees $D > {3 \over 2}n^{1/2}$ for large $n$.

For comparison let $K$ be the number of dots in an $n \times n$ array in which the vectors determined by different pairs of dots differ in length. As proved in [4] an upper bound for $K$ is

$$K < cn(\log n)^{-1/4},$$

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and a lower bound is

\[ K > n^{(2-\varepsilon)/3} \]

for any \( \varepsilon > 0 \) and sufficiently large \( n \).

**Theorem 1.** With distinct slopes, an upper bound for the number of dots is \( O(n^{4/5}) \).

**Proof.** Let us consider a set of \( D \) points in \( n \times n \) array such that all the slopes are distinct. Choose an integer \( m \) and write

\[ A_d = \{(id, jd) : 0 \leq i, j \leq m - 1\}, \]

a set of \( m^2 \) points. We shall use \( d \) as a variable. We cover the \( n \times n \) square by translated copies of \( A_d \). Since \( d^2 \) copies cover a square of size \( dm \times dm \), then

\[ d^2 \left( 1 + \frac{n}{dm} \right)^2 \leq 4n^2/m^2 \]

copies cover the whole square if \( dm \leq n \), that is,

\[ d \leq \left\lfloor \frac{n}{m} \right\rfloor. \tag{1} \]

If \( t \) denotes the number of copies used and \( w_1, \ldots, w_t \) are the number of points in each, then we have \( \sum w_i = D \), hence

\[ P = \sum \left( \begin{pmatrix} w_i \\ 2 \end{pmatrix} \right) \geq \frac{D(D-t)}{2t} \geq D^2/(4t) \geq \frac{1}{16} \frac{D^2m^2}{n^2} \]

if \( t \leq D/2 \), that is, if

\[ m^2 \geq 8n^2/D. \tag{2} \]

Each of the \( P \) pairs determines a vector whose coordinates are divisible by \( d \). Divide each vector by \( d \); we get a vector with coordinates \( < m \), altogether \( m^2 \) possibilities, and no two such vectors coincide, because that would mean two parallel vectors in the original system.

We have found \( \frac{D^2m^2}{16n^2} \) vectors for every \( d \), and \( d \) can run up to \( \left\lfloor n/m \right\rfloor \), so we have

\[ \frac{D^2m^2}{16n^2} \left\lfloor \frac{n}{m} \right\rfloor \leq m^2, \]

that is,

\[ D^2 \left\lfloor n/m \right\rfloor \leq 16n^2. \tag{3} \]

To optimize this, we select the maximal \( m \) allowed by (2):

\[ m = 1 + \left\lceil \sqrt{8n^2/D} \right\rceil, \quad \left\lfloor n/m \right\rfloor \geq \sqrt{D}/3 \]

for large \( n \). (3) yields \( D^{5/2} \leq 48n^2 \), hence

\[ D \leq 5n^{4/5}. \]
Theorem 2. It is possible to select \((1/2 + o(1))n^{1/2}\) grid points from an \(n \times n\) grid so that all slopes determined by pairs of points are distinct.

Proof. Let the grid consist of the integer points \((x, y), 0 \leq x, y < n\). Let \(q\) be the largest prime power so that \(m := q^2 + q + 1 < n\), and let \(p\) be the largest prime less than \(n\). The prime number theorem guarantees that \(q \sim \sqrt{n}, p \sim n\). By a classical result of Singer [1], there exists a perfect difference set \(A = \{a_1, a_2, \ldots, a_{q+1}\}\) of residues modulo \(m\), i.e., all the \(q^2 + q\) differences \(a_i - a_j, i \neq j\), are distinct modulo \(m\). Of course, if we translate \(A\) by forming \(A + d = \{a_i + d \mod m | a_i \in A\}\), then \(A + d\) is also a perfect difference set. As \(d\) runs from 1 to \(m\) the average of \(|\{A + d\} \cap [0, p/2]\|\) is exactly \((p + 1)(q + 1)/2m\). Therefore for some \(d\) at least \(p(q + 1)/2m\) of the elements of \(A + d\) lie in the interval \([0, p/2]\). Call the set of these elements \(B = \{b_1, \ldots, b_t\}\). Thus, \(0 < b_1 < \cdots < b_t < p/2\) where \(t \geq 2m(q + 1)/p(q + 1)\). Since all \(b_i - b_j, i \neq j\), are distinct \((\mod m)\), then all sums \(b_i + b_j, i \leq j\), are distinct \((\mod m)\), and consequently distinct as integers. Furthermore, since all \(b_i < p/2\) then all sums \(b_i + b_j, i \leq j\), are distinct \((\mod p)\). Let \(b_i^2 \equiv c_i \pmod{p}, 1 \leq i \leq t\).

Thus, \(b_i^2 = c_i + k_ip\) for integers \(k_i\), where \(0 \leq c_i < p\).

Now, for our set \(S\) of grid points we take

\[S = \{(b_i, c_i) | 1 \leq i \leq t\}.\]

(We have wrapped the parabola \(y = x^2\) around a \(p \times p\) torus and selected \(t\) points \((b_i, b_i^2)\) on it.) To check that \(S\) has the distinct slope property, we calculate for \(i < j\),

\[
\frac{c_i - c_j}{b_i - b_j} = \frac{b_i^2 - k_i p - (b_j^2 - k_j p)}{b_i - b_j} = \frac{b_i^2 - b_j^2}{b_i - b_j} - p \frac{b_i - b_j}{b_i - b_j} = \equiv b_i + b_j \pmod{p}
\]

since \(p\) is prime and \(b_i - b_j \neq 0 \pmod{p}\). Since by construction all the \(b_i + b_j, i < j\), are distinct, the \(S\) has the distinct slope property. Finally, since \(t \geq \frac{p(q + 1)}{2m} = (1/2 + o(1))q = (1/2 + o(1))\sqrt{n},\) then we are done.

It may be worth noting, in connection with Theorem 2, that if there are infinitely many primes \(q\) such that \(q^2 + q + 1\) is a prime (everybody believes this to be so) then there are infinitely many \(n\) for which it is possible to select \((1 + o(1))n^{1/2}\) points of an \(n \times n\) grid so that different pairs of points determine different slopes.

It would be interesting to know if we could actually get sets of size \(n^{1/2+\varepsilon}\) for a fixed \(\varepsilon > 0\).

Theorem 3. For large \(n\) suppose an \(n \times n\) array has \(D\) dots with all slopes distinct. If there are no points at which a new dot can be placed without causing a repeated slope, then \(D > \frac{3}{2}n^{1/2}\).

Proof. We will show that if \(D \leq \frac{3}{2}n^{1/2}\), then some of the \(n^2\) points will remain not excluded.
Each of the \(\left(\frac{D}{2}\right)\) distinct slopes is determined by a coprime pair of integers \(\{a, i\}\), and we may suppose \(a \leq i\), giving slopes \(\frac{a}{i}, \frac{i}{a}, \frac{-a}{i}, \frac{-i}{a}\). A line having one of these slopes, going through one of the \(D\) dots, will hit at most \(\frac{n}{i}\) other points of the \(n \times n\) array. Clearly smaller values of \(i\) produce more hits. For each \(i\) the number of coprime pairs with \(a \leq i\) is \(\phi(i)\), therefore the number of hits per dot will be less than \(\sum_{i=1}^{x} 4 \cdot \phi(i) \cdot \frac{n}{i}\) provided we choose \(x\) large enough to make \(\sum_{i=1}^{x} 4\phi(i) > \left(\frac{D}{2}\right)\).

To choose \(x\) we rely on the fact from [6] that \(\sum_{i=1}^{x} \phi(i) > \frac{3}{10}x^2\) for large \(x\). Accordingly, let \(x = \left\lceil \left(\frac{5}{12}\right)^{1/2} D \right\rceil\), and observe that

\[
\sum_{i=1}^{x} 4\phi(i) > \frac{12}{10} \cdot \frac{5}{12}D^2 = \frac{D^2}{2} > \left(\frac{D}{2}\right).
\]

Finally, using \(D \leq \frac{3}{5}n^{1/2}\), \(x < \frac{13}{20}D\), and \(\sum_{i=1}^{x} \phi(i) < x\), we obtain:

\[
D \cdot 4n \sum_{i=1}^{x} \frac{\phi(i)}{i} < \frac{3}{5}n^{1/2} \cdot 4n \cdot \frac{13}{20} \cdot \frac{3}{5}n^{1/2} = \frac{117}{125}n^2.
\]

Thus, counting dots plus hits, the total number of points excluded is less than \(\frac{3}{5}n^{1/2} + \frac{117}{125}n^2\), which is less than \(n^2\) for large \(n\).

**Theorem 4.** If an \(n \times m\) sonar sequence exists then \(m < n + 5n^{2/3}\).

**Proof.** Consider an \(n \times m\) sonar sequence. The array of dots and blanks has \(n\) rows and \(m\) columns, with one dot per column.

Let copies of an \(R \times R\) window be translated so that each dot or blank sits in \(R^2\) windows. Thus the number of windows used will be \(W = (n + R - 1)(m + R - 1)\). The average number of dots per window is \(A = \frac{R^2m}{W}\).

If the \(i^{th}\) window has \(w_i\) dots, then the number of occurrences of a pair of dots in a window is

\[
\sum_{i=1}^{W} \frac{w_i(w_i - 1)}{2}.
\]

Since \(A\) is the average,

\[
\frac{WA(A - 1)}{2} \leq \sum_{i=1}^{W} \frac{w_i(w_i - 1)}{2}.
\]

Because pairs of dots determine distinct vectors, we can get an upper bound on the actual number of occurrences of a pair of dots in a window by counting all the possible patterns of two dots in a window, as follows. There are \(R^2\) places to put the first dot, then, allowing only one per column, there are \(R(R - 1)\) places to put the second dot. That counts each twice, so the number of possible patterns of two dots in a window is \(\frac{R^2R(R - 1)}{2}\).
Putting these restrictions together we have:

\[
\frac{WA(A-1)}{2} \leq \sum_{i=1}^{W} \frac{w_i(w_i-1)}{2} \leq \frac{R^2(R^2 - R)}{2},
\]

\[
m(A - 1) \leq R^2 - R,
\]

\[
\frac{mr^2m}{W} \leq R^2 + m - R,
\]

\[
m \leq \frac{W}{m} + \frac{W}{R^2} - \frac{W}{Rm}.
\]

Expanding this we choose the integer \(R\) such that \(n^{2/3} \leq R < n^{2/3} + 1\), so that \(W < nm + n^{2/3}m + n^{5/3} + n^{4/3}\), and \(R^2 \geq n^{4/3}\). From these we can see that \(m < n + n^{2/3} + \frac{n^{5/3} + n^{4/3}}{m} + \frac{m}{n^{1/3}} + \frac{m}{n^{2/3}} + n^{1/3} + 1\). Finally because \(n < m < 2n\) we have \(m < n + 4n^{2/3} + 4n^{1/3} + 1\). Thus the proof is complete that for large \(n\), \(m < n + 5n^{2/3}\).

\[\square\]

**Comment.** More careful computation shows that actually \(m < n + 3n^{2/3} + 2n^{1/3} + 9\) for all \(n\).

**Theorem 5.** For some constant \(c > 0\) there are infinitely many integers \(n\) such that an \(n \times m\) sonar sequence exists with \(n > n + c \log n \log \log \log n\).

**Proof.** In [5] S. W. Graham and C. J. Ringrose prove that, for infinitely many primes \(p\), the least quadratic non-residue between 1 and \(p\) is larger than \(L = c \log p \log \log \log p\).

If \(p\) is any odd prime, a \(p \times p\) sonar sequence \(a_1, a_2, \ldots, a_p\) can be obtained by letting \(a_i \equiv bi^2 \pmod{p}\) where \(b \not\equiv 0 \pmod{p}\), and choosing \(1 \leq a_i \leq p\). To verify that distinct pairs of dots determine distinct vectors notice modulo \(p\) that \(a_{i+k} - a_i \equiv a_{j+k} - a_j\) only if \(2ik \equiv 2jk\), which happens only when \(k \equiv 0\) or \(i \equiv j\).

When \(p\) is such that all the numbers between 1 and \(L\) are quadratic residues, and \(b\) is a non-residue, we find \(L < a_i \leq p\) for each \(i\) from 1 to \(p\). Thus with \(m = p\), and \(n < p - L\), this will be an \(n \times m\) sonar sequence with \(m > n + c \log n \log \log \log n\).

\[\square\]

**Open Problem** Does an \(n \times n\) array with \(n\) dots exist for every \(n\), in which distinct pairs of dots determine vectors which differ in slope or in length?

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**References**


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