Quasi-random graphs with given degree sequences

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Abstract

It is now known that many properties of the objects in certain combinatorial structures are equivalent, in the sense that any object possessing any of the properties must of necessity possess them all. These properties, termed quasirandom, have been described for a variety of structures such as graphs, hypergraphs, tournaments, Boolean functions, and subsets of $\mathbb{Z}_n$, and most recently, sparse graphs. In this paper, we extend these ideas to the more complex case of graphs which have a given degree sequence.

1 Introduction

During recent years there has been increasing interest in investigating the following phenomenon. For a given finite collection $C$ of "objects", suppose we have some probability distribution given on $C$. Typically, there are many properties which are satisfied by most (or almost all) of the objects in $C$ as seen in [4]. It turns out, however, that in many cases there is a large subclass $Q$ of these properties which are strongly correlated, in the sense that any object in $C$ which satisfies any of the properties in $Q$ must in fact necessarily satisfy all the properties in $Q$. Such properties are called "quasi-random". Specific cases where this behavior is investigated can be found in [14, 15, 20] (for graphs), [11, 12, 16, 18] (for hypergraphs), [13] (for tournaments), [17] (for sequences), [24] (for permutations) and [19] (for sparse graphs), for example.

In this paper we will take $C$ to be the class $G_n(d)$ of all graphs on $n$ vertices having some given degree sequence $d$. This is rather different from the classical model of a random graph, in which all vertices have the same expected degree. Special cases of such graph families include the so-called power law graphs in which the number of vertices of degree $k$ is proportional to $k^{-\beta}$ for some positive

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real $\beta$. Such graphs arise in a variety of applications such as Web connectivity
[5, 6, 9, 23, 25, 27, 28], communication networks [1, 3], biological networks [21],
collaboration graphs [26], etc.

In this paper, we will introduce a class of quasi-random properties for $G_n(d)$
and establish quantitative bounds on the strength of correlation between these
properties. In particular, these results generalize and strengthen those in [19,
20].

2 Notation

We will consider graphs $G = (V, E)$ where $V$ denotes the set of vertices of $G$
and $E$ denotes the set of edges of $G$. (For undefined graph theory terminology,
see [29].) Our graphs will be undirected, having no loops or multiple edges. We
will let $|V|$, the cardinality of $V$, be denoted by $n$.

If $\{x, y\} \in E$ is an edge of $G$, we say that $x$ and $y$ are adjacent, and write
this as $x \sim y$. The neighborhood $nd(x)$ of a vertex $x \in V$ is defined by

$$nd(x) := \{y \in V : y \sim x \text{ in } G\}.$$  

For $x \in V$, the degree $d_x$ of $x$, denotes $|nd(x)|$. The degree sequence $d = d_G$ of
$G$ is given by

$$d = (d_x : x \in V),$$
or equivalently, $d$ can be viewed as a mapping $d : V \rightarrow \mathbb{Z}^+ \cup \{0\}$. For $X, Y \subseteq V$,
define

$$e(X, Y) := |\{(x, y) : x \in X, y \in Y \text{ and } x \sim y\}|.$$  

For $X \subseteq V$, define $\text{vol}(X)$, the volume of $X$, by

$$\text{vol}(X) = \sum_{x \in X} d_x.$$  

A walk $P = P_t(x, y)$ from $x$ to $y$ is a sequence $P = (x_0, x_1, \ldots, x_t)$, where
$x_0 = x, x_t = y$ and $x_i \sim x_{i+1}$ for $0 \leq i < t$. Such a walk is said to have length
$t$. Here we do not require all $x_i$’s to be distinct. If all $x_i$’s are different, we say
the walk is a path.

The weight $w(P)$ of such a walk $P$ is defined to be

$$w(P) = \prod_{0 \leq i < t} \frac{1}{d_{x_i}}$$
(thus, both endpoints are excluded in the product). If $P$ has length 1 (and
therefore is an edge of $G$), then $w(P)$ is defined to be 1.
A circuit $C$ of length $t$ is a sequence of $t$ vertices $(x_1, x_2, \ldots, x_t)$ where $x_i \sim x_{i+1}, 1 \leq i < t$, and $x_t \sim x_1$. (We remark that in this definition, a circuit can be viewed as a rooted closed walk.) The weight $w(C)$ of such a circuit is defined by

$$w(C) = \prod_{1 \leq i \leq t} \frac{1}{d_{x_i}}.$$ 

The weighted adjacency matrix $M = M(G)$ is an $n \times n$ matrix with rows and columns indexed by $V$, with

$$M(x, y) = \begin{cases} \frac{1}{\sqrt{d_x d_y}} & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $M$ are denoted by $\rho_i, 0 \leq i \leq n - 1$, indexed so that

$$1 = \rho_0 \leq |\rho_1| \geq |\rho_2| \geq \ldots \geq |\rho_{n-1}|.$$ 

Note that $\rho_0 = 1$ has as its eigenvector $(\sqrt{d_x})_{x \in V}$.

Finally, define for $X, Y \subseteq V$, and $t \geq 1$,

$$e_t(X, Y) = \sum_{P \in P_t(X, Y)} w(P)$$

where $P_t(X, Y)$ denotes the set of all walks of length $t$ between $x \in X$ and $y \in Y$. This is a weighted version of the number of walks of length $t$ between $X$ and $Y$. Note that $e_1(X, Y) = e(X, Y)$.

### 3 The quasi-random properties

In this section we will state various properties that the $G \in \mathcal{G}_n(d)$ might satisfy. Each of these properties will depend on a parameter $\epsilon$, which we will always assume to satisfy $0 < \epsilon < 1$. The closer $\epsilon$ is to $0$, the more the graph in question behaves like a random graph with respect to the property in question, that is, the more the value of the corresponding parameter is closer to its expected value for a random graph in $\mathcal{G}_n(d)$.

**DISC(\epsilon):**

For all $X, Y \subseteq V$,

$$|e(X, Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)}| \leq \epsilon \text{vol}(G).$$
**DISC}_t(\epsilon):**

For all $X, Y \subseteq V$,

$$|e_t(X, Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)}| \leq \epsilon \text{ vol}(G).$$

Note that DISC}_1(\epsilon) is just DISC(\epsilon).

**EIG}(\epsilon):**

With the matrix $M = M(G) = (M(x, y))_{x, y \in V}$ defined as before, i.e.,

$$M(x, y) = \begin{cases} \\ \frac{1}{\sqrt{d_x d_y}} & \text{if } x \sim y, \\ 0 & \text{otherwise,} \end{cases}$$

and with eigenvalues satisfying

$$1 = \rho_0 \geq |\rho_1| \geq |\rho_2| \geq \ldots \geq |\rho_{n-1}|,$$

we have

$$|\rho_i| < \epsilon \quad \text{for all } i \geq 1.$$

**TRACE}_t(\epsilon):**

The eigenvalues of $M$ satisfy

$$\sum_{i \geq 1} \rho_i^{2t} \leq \epsilon.$$

**CIRCUIT}_t(\epsilon):**

The weighted sum of the $t$-circuits $C_t$ in $G$ satisfies

$$\left| \sum_{C_t: \text{t-circuit}} w(C_t) - 1 \right| \leq \epsilon.$$

### 4 The implications

**Lemma 1** EIG(\epsilon) $\implies$ DISC(\epsilon).

**Proof:** For $S \subseteq V$, define
\[ f_S(x) = \begin{cases} \sqrt{d_x} & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases} \]

Then, for \( X, Y \subseteq V \),
\[ e(X, Y) = \langle f_X, M f_Y \rangle \]
where \( \langle f, g \rangle = \sum_{x \in V} f(x)g(x) \) denotes the usual inner product.

Now, write
\[ f_X = \sum_i a_i \phi_i \]
where the \( \phi_i \)'s form an orthonormal basis of eigenvectors with
\[ \phi_0(v) = \sqrt{\frac{d_v}{\text{vol}(G)}}, \]
for all \( v \in V \). Hence,
\[ a_0 = \langle f_X, \phi_0 \rangle = \sum_{x \in X} \frac{d_x}{\sqrt{\text{vol}(G)}} = \frac{\text{vol}(X)}{\sqrt{\text{vol}(G)}} \]

Similarly, we have the expansion
\[ f_Y = \sum_i b_i \phi_i. \]

Thus,
\[ \langle f_X, M f_Y \rangle = a_0 b_0 + \sum_{i \geq 1} \rho_i a_i b_i = \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} + \sum_{i \geq 1} \rho_i a_i b_i \]

Therefore,
\[ \left| e(X, Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} \right| = \left| \sum_{i \geq 1} \rho_i a_i b_i \right| \leq \max_{i \geq 1} |\rho_i| \left( \sum_{i \geq 1} |a_i|^2 \right)^{1/2} \left( \sum_{i \geq 1} |b_i|^2 \right)^{1/2} \leq \epsilon \|f_X\| \|f_Y\| = \epsilon \sqrt{\text{vol}(X)\text{vol}(Y)} \leq \epsilon \text{ vol}(G) \]
and the proof is complete.

In a similar way, we prove

**Lemma 2** \( \text{EIG}(\varepsilon) \Rightarrow \text{DISC}_t(\epsilon^t) \) for any \( t \geq 1 \).

**Proof:** In this case we observe that for \( X, Y \subseteq V \),
\[
\epsilon_t(X, Y) = \langle f_X, M^t f_Y \rangle
\]
(using the notation of Lemma 1). Thus, writing
\[
f_X = \sum_i a_i \phi_i, f_Y = \sum_i b_i \phi_i,
\]
we find
\[
\frac{|\langle f_X, M^t f_Y \rangle - \text{vol}(X)\text{vol}(Y)|}{\text{vol}(G)} = |\langle f_X, M^t f_Y \rangle - \rho_0 a_0 b_0|
\]
\[
\leq \max_{i \geq 1} |\rho_i| \sum_{i \geq 1} |a_i b_i|
\]
\[
\leq \max_{i \geq 1} |\rho_i| \|f_X\| \|f_Y\|
\]
\[
\leq \epsilon^t \sqrt{\text{vol}(X)\text{vol}(Y)}
\]
\[
\leq \epsilon^t \text{vol}(G)
\]
and Lemma 2 is proved.

**Lemma 3** \( \text{CIRCUIT}_{2t}(\epsilon) \iff \text{TRACE}_{2t}(\epsilon) \).

**Proof:** Let \( C_{2t}^*(u) \) denote a rooted \( 2t \)-circuit with starting and ending point \( u \). Then,
\[
M^{2t}(u, u) = \sum_{C_{2t}^*(u)} w(C_{2t}^*(u)).
\]
Thus,
\[
\text{Tr}(M^{2t}) = \sum_u \sum_{C_{2t}^*(u)} w(C_{2t}^*(u)) = \sum_{C_{2t}} w(C_{2t}).
\]
On the other hand,
\[
\text{Tr}(M^{2t}) = \sum_i \rho_i^{2t} = 1 + \sum_{i \geq 1} \rho_i^{2t}.
\]
Thus, we have
\[ | \sum_{C_{2t}} w(c_{2t}) - 1 | = | \text{Tr}(M^{2t}) - 1 | \]
\[ = \sum_{i \geq 1} \rho_i^{2t} \]
and Lemma 3 is proved. \qed

**Lemma 4** \( \text{TRACE}_{2t}(\epsilon) \implies \text{EIG}(\epsilon^{1/2t}), \text{for any } t \geq 1. \)

**Proof:** By hypothesis, we have
\[ | \sum_{i} \rho_i^{2t} - 1 | = \sum_{i \geq 1} \rho_i^{2t} \leq \epsilon. \]
Therefore
\[ \max_{i \geq 1} | \rho_i | \leq \epsilon^{1/2t}. \]
\qed

**Lemma 5** \( \text{TRACE}_{2t}(\epsilon) \implies \text{TRACE}_{2t+2}(\epsilon). \)

**Proof:** By Perron-Frobenius, \( | \rho_i | \leq 1 \) for all \( i \). Thus
\[ \sum_{i \geq 1} \rho_i^{2t+2} \leq \sum_{i \geq 1} \rho_i^{2t} \leq \epsilon \]
by hypothesis. \qed

**Lemma 6** \( \text{For } t \geq 1, \text{DISC}_{2t}(\epsilon) \implies \text{DISC}_t(\sqrt{\epsilon}). \)

**Proof:** For \( X \subseteq V \),
\[ e_{2t}(X, X) = \sum_{x, x' \in X} \sum_y \frac{e_t(x, y)e_t(y, x')}{d_y} \]
\[ = \sum_y \frac{e_t(y, X)^2}{d_y}. \]
From \( \text{DISC}_{2t}(\epsilon) \), we have
\[ \sum_y \frac{e_t(y, X)^2}{d_y} \leq \frac{\text{vol}(X)^2}{\text{vol}(G)} + \epsilon \text{vol}(G). \]
Note that
\[ \sum_y e_t(y, X) = e_t(V, X) = \text{vol}(X). \]

Therefore,
\[
\begin{align*}
\sum_y (e_t(y, X) - \frac{d_y \text{vol}(X)}{\text{vol}(G)})^2 \frac{1}{d_y} &= \sum_y e_t(y, X)^2 - 2e_t(V, X) \frac{\text{vol}(X)}{\text{vol}(G)} + \frac{\text{vol}(X)^2}{\text{vol}(G)} \\
&= \sum_y e_t(y, X)^2 - \frac{\text{vol}(X)^2}{\text{vol}(G)} \\
&\leq \epsilon \text{ vol}(G).
\end{align*}
\]

by DISC\(_{2t}(\epsilon)\). But
\[
\begin{align*}
\sum_y (e_t(y, X) - \frac{d_y \text{vol}(X)}{\text{vol}(G)})^2 \frac{1}{d_y} &\geq \sum_{y \in Y} (e_t(y, X) - \frac{d_y \text{vol}(X)}{\text{vol}(G)})^2 \frac{1}{d_y} \\
&\geq \left( \sum_{y \in Y} (e_t(y, X) - \frac{d_y \text{vol}(X)}{\text{vol}(G)}) \right)^2 \frac{1}{\sum_{y \in Y} d_y} \\
&\geq \left( e_t(Y, X) - \frac{\text{vol}(Y)\text{vol}(X)}{\text{vol}(G)} \right)^2 /\text{vol}(Y).
\end{align*}
\]

Thus,
\[
| e_t(Y, X) - \frac{\text{vol}(Y)\text{vol}(X)}{\text{vol}(G)} | \leq \sqrt{\epsilon \text{ vol}(Y)\text{vol}(G)} \\
\leq \sqrt{\epsilon \text{ vol}(G)}.
\]

This is exactly DISC\(_t(\sqrt{\epsilon})\). \hfill \square

**Lemma 7** For any \( t \geq 1 \), DISC\(_t(\epsilon) \implies \text{DISC}_{t+1}(6\sqrt{\epsilon}).

**Proof:**

For \( X, Y \subseteq V \),
\[
| e_t(X, Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} | \leq \epsilon \text{ vol}(G). \tag{1}
\]
Consider
\[ e_{t+1}(X, Y) = \sum_v e(X, v)e_t(v, Y) \cdot \frac{d_v}{d_v}. \]

Define
\[ S_1 := \{ z \in V : e_t(z, Y) > \frac{d_z}{\text{vol}(G)} (\text{vol}(Y) + \sqrt{\epsilon} \text{vol}(G)) \}. \]

Thus,
\[ \sum_{z \in S_1} e_t(z, Y) = e_t(S_1, Y) > \frac{\text{vol}(S_1)\text{vol}(Y)}{\text{vol}(G)} + \sqrt{\epsilon} \text{vol}(S_1). \]

Hence, by (1),
\[ \text{vol}(S_1) < \sqrt{\epsilon} \text{vol}(G). \]

In the same way, if we define
\[ S_2 := \{ z \in V : e_t(z, Y) < \frac{d_z}{\text{vol}(G)} (\text{vol}(Y) - \sqrt{\epsilon} \text{vol}(G)) \} \]
then
\[ \text{vol}(S_2) < \sqrt{\epsilon} \text{vol}(G). \]

Now,
\[ e_{t+1}(X, Y) = \left( \sum_{v \in S_1 \cup S_2} + \sum_{v \in S_1 \cup S_2} \right) e(X, v)e_t(v, Y) \cdot \frac{d_v}{d_v}. \]

For the first sum, we have
\[ \sum_{v \in S_1 \cup S_2} \frac{e(X, v)e_t(v, Y)}{d_v} \leq \sum_{v \in S_1 \cup S_2} \frac{e(X, v)}{d_v} \frac{d_v}{\text{vol}(G)} (\text{vol}(Y) + \sqrt{\epsilon} \text{vol}(G)) \]
\[ \leq \sum_{v} \frac{e(X, v)}{\text{vol}(G)} (\text{vol}(Y) + \sqrt{\epsilon} \text{vol}(G)) \]
\[ \leq \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} + \sqrt{\epsilon} \text{vol}(G) \]

and
\[ \sum_{v \notin S_1 \cup S_2} \frac{e(X, v)e_t(v, Y)}{d_v} \geq \sum_{v \notin S_1 \cup S_2} \frac{e(X, v)}{\text{vol}(G)} (\text{vol}(Y) - \sqrt{\epsilon} \text{vol}(G)) \]
\[ \geq \sum_{v \notin S_1 \cup S_2} \frac{e(X, v)}{\text{vol}(G)} (\text{vol}(Y) - \sqrt{\epsilon} \text{vol}(G)) \]
\[ \geq \frac{(\text{vol}(X) - \text{vol}(S_1) - \text{vol}(S_2))}{\text{vol}(G)} (\text{vol}(Y) - \sqrt{\epsilon} \text{vol}(G)) \]
\[ \geq \frac{(\text{vol}(X) - 2\sqrt{\epsilon} \text{vol}(G))}{\text{vol}(G)} (\text{vol}(Y) - \sqrt{\epsilon} \text{vol}(G)) \]
\[ \geq \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} - 3\sqrt{\epsilon} \text{vol}(G). \]
Thus,
\[ | \sum_{v \in S_1 \cup S_2} \frac{e(X,v)e_t(v,Y)}{d_v} - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} | \leq 3\sqrt{\epsilon} \text{vol}(G). \]

For the second sum, we have
\[ \sum_{v \in S_1 \cup S_2} \frac{e(X,v)e_t(v,Y)}{d_v} \leq \sum_{v \in S_1 \cup S_2} \frac{d_v e_t(v,Y)}{d_v} = e_t(S_1 \cup S_2, Y) \leq \frac{(\text{vol}(S_1) + \text{vol}(S_2))\text{vol}(Y)}{\text{vol}(G)} + \epsilon \text{vol}(G) \text{ by DISC}_t(\epsilon) \leq 2\sqrt{\epsilon} \frac{\text{vol}(G)\text{vol}(Y)}{\text{vol}(G)} + \epsilon \text{vol}(G) = 2\sqrt{\epsilon} \text{vol}(Y) + \epsilon \text{vol}(G) \leq 3\sqrt{\epsilon} \text{vol}(G). \]

Putting these two estimates together, we obtain
\[ | e_{t+1}(X,Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} | \leq 6\sqrt{\epsilon} \text{vol}(G) \]
which is DISC\(_{t+1}(6\sqrt{\epsilon})\).

\[ \square \]

**Lemma 8** For any integers \( s \) and \( t \), DISC\(_s\) and DISC\(_t\) are related as follows:

(i) If \( s < t \), then DISC\(_s\)(\(\epsilon\)) \(\Rightarrow\) DISC\(_t\)(\(36\epsilon^{1/2^{t-s}}\)).
As a special case, DISC(\(\epsilon\)) \(\Rightarrow\) DISC\(_t\)(\(36\epsilon^{1/2^{t-1}}\)).

(ii) If \( t < s \leq 2^k t \) for some \( k \), then DISC\(_s\)(\(\epsilon\)) \(\Rightarrow\) DISC\(_t\)(\(36\epsilon^{1/2^k} \epsilon^{1/2^{k+s-k}}\)).

Proof: (i) follows from Lemma 7, i.e.,
\[
\text{DISC}_s(\epsilon) \Rightarrow \text{DISC}_s(36\epsilon) \\
\Rightarrow \text{DISC}_{s+1}(36\epsilon^{1/2}) \\
\Rightarrow \ldots \\
\Rightarrow \text{DISC}_t(36\epsilon^{1/2^{t-s}})
\]

To prove (ii), we have, from (i) that
\[
\text{DISC}_s(\epsilon) \Rightarrow \text{DISC}_{2^k t}(36\epsilon^{1/2^{k+t-s}}) 
\]
Now apply Lemma 6 \( k \) times to get the desired implication.

We can summarize the preceding results as follows:
Theorem 1 For \( t \geq 2 \), the following implications hold.

\[
\begin{align*}
\text{CIRCUIT}_{2^t} & \iff \text{TRACE}_{2^t} \quad \epsilon^{1/2^t} & \iff \text{EIG} & \iff \text{DISC} \quad \sqrt{2} & \iff \text{DISC}_2 \\
\Downarrow \ & \epsilon \quad & \preceq \epsilon^t \quad & \Downarrow 36e^{1/2^t - 1} \quad & \text{TRACE}_{2^{t+2}} \quad & \text{DISC}_t
\end{align*}
\]

Here the notation \( A \overset{\delta}{\Rightarrow} B \) is shorthand for \( A(\epsilon) \Rightarrow B(\delta) \).

5 Separation of properties

In this section we give an example showing that at least one of the implications in Theorem 1 cannot be reversed. Whether this is true of the others is not known at this point.

Fact 1 For any \( t \geq 1 \),

\[
\text{EIG}(\epsilon) \not\iff \text{TRACE}_{2^t}(\delta)
\]

for any \( \delta = \delta(\epsilon) \).

Proof: Choose \( t \geq 1 \) and let \( G = G(n) \) be a random regular graph with \( n \) vertices and vertex degree \( n^{1/t} \). Thus, \( M = M(G) \) has

\[
M(u, v) = \begin{cases} 
1/n^{1/t} & \text{if } u \sim v, \\
0 & \text{otherwise}.
\end{cases}
\]

It was shown in [22] that the eigenvalue distribution of \( M(G) \) for a random graph \( G \) with a given expected degree distribution satisfies the semi-circle law. As a consequence, if \( 1 = \rho_0 \geq |\rho_1| \geq |\rho_2| \geq \ldots \geq |\rho_{n-1}| \) are the eigenvalues of \( M \), then

\[(1) \quad \rho_1 = (1 + o(1))2/n^{1/2t}, \]

\[(2) \quad \text{If } N(x) \text{ denotes the number of } \rho_i \text{ with } \rho_i \leq 2x/n^{1/2t}, \text{ then} \]

\[
\frac{N(x)}{n} = (1 + o(1))\frac{2}{\pi} \int_{-1}^{x} \sqrt{1 - u^2}du.
\]
In particular, for $x = 1/2$, we have
\[
\frac{N(1/2)}{n} = (1 + o(1))(\frac{2}{3} + \frac{\sqrt{3}}{4\pi}) \approx 0.8045 \ldots
\]

Thus,
\[
\sum_{i \geq 1} \rho_i^{2t} \geq 2(n - N(1/2))(\frac{1}{n^{1/2t}})^{2t} \geq 0.391.
\]

Hence, for any $\epsilon > 0$, $G$ satisfies EIG($\epsilon$), provided $n \geq n_0$, but $G$ does not satisfy TRACE$_{2t}(0.39)$.

It would be interesting to know if some of the other possible implications hold. For example, does DISC $\Rightarrow$ EIG?

Recently, Bilu and Linial [7] proved the following partial implication for regular graphs:

For a $d$-regular graph $G$ on $n$ vertices, if for all $X, Y \subset V$,
\[
|e(X,Y) - \frac{d}{n} |X||Y|| \leq \alpha d \sqrt{|X||Y|},
\]

then $|\rho_1| = O(\alpha(\log(1/\alpha) + 1))$.

Butler [10] combines the methods in [7] and [8] to prove the following:

For a graph $G$ with no isolated vertices, if for all $X, Y \subset V$,
\[
|e_t(X,Y) - \frac{\operatorname{vol}(X)\operatorname{vol}(Y)}{\operatorname{vol}(G)}| \leq \alpha \sqrt{\operatorname{vol}(X)\operatorname{vol}(Y)},
\]

then $|\rho_1|^t \leq 18\alpha(1 - \frac{5}{2} \log \alpha)$.

For $t = 1$, this is the best possible (up to a constant) by considering a class of regular graphs constructed by Bollobás and Nikiforov [8]. In their example, the graphs have $\alpha = Cn^{-1/6}$ and $|\rho_1| \geq c \alpha \log n$ for some constants $c$ and $C$.

6 Reversing the implications

It is clear from the examples in the preceding section that in order to establish some of the reverse implications, e.g., DISC $\Rightarrow$ CIRCUIT$_{2t}$, we will have to make further assumptions for the $G \in G_n(d)$. One such condition is the following:

For $t \geq 1$, a graph satisfies $U_t(C)$ if for all $x, y \in V, e_t(x,y) \leq C \frac{d_x d_y}{\operatorname{vol}(G)}$.

We will think of $C$ as a large positive real. We note that for $t = 1$ and for $G$ with minimum degree $\alpha n$, the property $U_1(C)$ is automatically satisfied for $C \geq 1/\alpha^2$. 

Note that for a $d$-regular graph, $U_t$ implies that $n \leq Cd^t$ or, equivalently, the volume of the graph is of order at least $n^{1+1/t}$.

**Lemma 9** For any $t \geq 1$,

$$U_t(C) \implies U_{t+1}(C).$$

**Proof:** Observe that

$$e_{t+1}(x, y) = \sum_z \frac{e(x, z)e_t(z, y)}{d_z} \leq \sum_z \frac{e(x, z)}{d_z} \cdot C \frac{d_x d_y}{\text{vol}(G)} = C \frac{d_y}{\text{vol}(G)} \sum_z e(x, z) = C \frac{d_x d_y}{\text{vol}(G)}.$$

The lemma is proved. $\Box$

**Theorem 2** If $G$ satisfies $U_{t-1}(C)$ for some $t \geq 2$, then

$$\text{DISC}(\epsilon) \implies \text{CIRCUIT}_{2t}(\eta)$$

where $\eta = 2C^2C^2\epsilon/\delta + 2C^2(C^2+1)^2\delta + 20\sqrt{\delta} + 12\delta + 16C^2\delta^{3/2} + 8C^2\delta^{3/2}$, with $C' = [C/\delta^{1/4}]$, and $\delta = \max\{\sqrt{\epsilon}, 36\epsilon^{1/2^{t-2}}\}$. (Note that $\eta \to 0$ as $\epsilon \to 0$.)

**Proof:** We are going to consider the sum

$$\sum_{u, v \in V} \frac{1}{d_x d_y} (e_t(u, v) - \frac{d_x d_y}{\text{vol}(G)})^2$$

where, as usual, $V = V(G)$.

Since $G$ satisfies DISC$(\epsilon)$ by hypothesis, then by Lemma 8, $G$ also satisfies DISC$_{t-1}(\delta)$ where $\delta \geq 36\epsilon^{1/2^{t-2}}$, i.e.,

$$|e_{t-1}(X, Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)}| \leq \delta \text{ vol}(G)$$

for all $X, Y \subseteq V$.

We here choose $\delta = \max\{\sqrt{\epsilon}, 36\epsilon^{1/2^{t-2}}\}$. We partition the vertex set $V$ into the
sets $W_i$, $0 \leq i < C'$, as follows:

\[
W_0 = \{ v : 0 \leq \epsilon_{t-1}(u, v) < \delta^{1/4} \frac{d_u d_v}{\text{vol}(G)} \},
\]

\[
W_1 = \{ v : \delta^{1/4} \frac{d_u d_v}{\text{vol}(G)} \leq \epsilon_{t-1}(u, v) < 2\delta^{1/4} \frac{d_u d_v}{\text{vol}(G)} \},
\]

\[
W_2 = \{ v : 2\delta^{1/4} \frac{d_u d_v}{\text{vol}(G)} \leq \epsilon_{t-1}(u, v) < 3\delta^{1/4} \frac{d_u d_v}{\text{vol}(G)} \},
\]

and, in general,

\[
W_i = \{ v : i\delta^{1/4} \frac{d_u d_v}{\text{vol}(G)} \leq \epsilon_{t-1}(u, v) < (i+1)\delta^{1/4} \frac{d_u d_v}{\text{vol}(G)} \}
\]

for $0 \leq i < C' = [C/\delta^{1/4}]$. Since $\epsilon_{t-1}(u, v) \leq C d_u d_v / \text{vol}(G)$ by $U_{t-1}(C)$, the $W_i$ form a partition of $V$.

Since

\[
W_i \subseteq \{ v : |\epsilon_{t-1}(u, v) - i\delta^{1/4} \frac{d_u d_v}{\text{vol}(G)}| < \delta^{1/4} \frac{d_u d_v}{\text{vol}(G)} \},
\]

then

\[ |\sum_{v \in W_i} (\epsilon_{t-1}(u, v) - i\delta^{1/4} \frac{d_u d_v}{\text{vol}(G)})| < \sum_{v \in W_i} \delta^{1/4} \frac{d_u d_v}{\text{vol}(G)} \]

and

\[ |\epsilon_{t-1}(u, W_i) - i\delta^{1/4} \frac{d_u \text{vol}(W_i)}{\text{vol}(G)}| < \delta^{1/4} \frac{d_u \text{vol}(W_i)}{\text{vol}(G)}. \tag{2} \]

Since $\sum_i \epsilon_{t-1}(u, W_i) = \epsilon_{t-1}(u, V) = d_u$, then

\[ |\sum_i i\delta^{1/4} \frac{d_u \text{vol}(W_i)}{\text{vol}(G)} - d_u| = |\sum_i i\delta^{1/4} \frac{d_u \text{vol}(W_i)}{\text{vol}(G)} - \sum_i \epsilon_{t-1}(u, W_i)| \leq \delta^{1/4} \sum_i \frac{d_u \text{vol}(W_i)}{\text{vol}(G)} = \delta^{1/4} d_u. \tag{3} \]

Now, for each $i$, if $\text{vol}(W_i) \geq \sqrt{\delta} \text{vol}(G)$, then define

\[
X_i = \{ v : e(v, W_i) > \frac{d_u \text{vol}(W_i)}{\text{vol}(G)} (1 + \sqrt{\delta}) \},
\]

\[
X'_i = \{ v : e(v, W_i) < \frac{d_u \text{vol}(W_i)}{\text{vol}(G)} (1 - \sqrt{\delta}) \}.
\]

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If $\text{vol}(W_i) < \sqrt{\delta}\text{vol}(G)$ then define $X_i = X_i' = \emptyset$. Also define
\[ W^* = \cup \{ W_i : \text{vol}(W_i) < \sqrt{\delta}\text{vol}(G) \}. \]
Thus,
\[ \text{vol}(W^*) \leq C'\sqrt{\delta}\text{vol}(G) \]
since there are just $C'$ possible values of $i$.

By DISC($\epsilon$), we have
\[ |e(W_i, X_i) - \frac{\text{vol}(W_i)\text{vol}(X_i)}{\text{vol}(G)}| \leq \epsilon \text{vol}(G), \]
but
\[
|e(W_i, X_i) - \frac{\text{vol}(W_i)\text{vol}(X_i)}{\text{vol}(G)}| \geq \sqrt{\delta} \frac{\text{vol}(X_i)\text{vol}(W_i)}{\text{vol}(G)} \\
\geq \sqrt{\delta} \frac{\sqrt{\delta}\text{vol}(G)\text{vol}(X_i)}{\text{vol}(G)} \\
= \delta \text{vol}(X_i). 
\]
Therefore,
\[ \text{vol}(X_i) \leq \epsilon/\delta \text{ vol}(G). \]
A similar argument shows that
\[ \text{vol}(X_i') \leq \epsilon/\delta \text{ vol}(G) \]
as well. Consequently, for
\[ X_u := \cup \{ X_i \cup X_i' : W_i \not\subseteq W^* \} \]
we have
\[ \text{vol}(X_u) \leq 2C'\epsilon/\delta \text{ vol}(G). \]  \hspace{1cm} (4)
For $v \not\in X_u$, we have, from the definition of $X_u$,
\[
e(W^*, v) = d_v - \sum_{W_i \not\subseteq W^*} e(W_i, v) \\
\leq d_v - \sum_{W_i \not\subseteq W^*} (1 - \sqrt{\delta}) \frac{d_v\text{vol}(W_i)}{\text{vol}(G)} \\
= d_v - (1 - \sqrt{\delta}) \frac{d_v(\text{vol}(G) - \text{vol}(W^*))}{\text{vol}(G)} \\
= \sqrt{\delta}d_v + (1 - \sqrt{\delta}) \frac{\text{vol}(W^*)d_v}{\text{vol}(G)} \\
\leq \sqrt{\delta}d_v + (1 - \sqrt{\delta})C'\sqrt{\delta}\text{vol}(G)d_v \\
\leq (C' + 1)\sqrt{\delta}d_v. \]  \hspace{1cm} (5)
We now begin considering the sum,
\[
\sum_u \sum_v \frac{1}{d_u d_v} (e_t(u, v) - \frac{d_u d_v}{\text{vol}(G)})^2 = \\
\sum_u \left( \sum_{v \in X_u} + \sum_{v \notin X_u} \right) \frac{1}{d_u d_v} (e_t(u, v) - \frac{d_u d_v}{\text{vol}(G)})^2.
\]
For the first sum, we have the estimate
\[
\sum_u \sum_{v \in X_u} \frac{1}{d_u d_v} (e_t(u, v) - \frac{d_u d_v}{\text{vol}(G)})^2 \leq \sum_u \sum_{v \in X_u} \frac{1}{d_u d_v} C^2 \left( \frac{d_u d_v}{\text{vol}(G)} \right)^2 = C^2 \frac{\text{vol}(X_u)}{\text{vol}(G)} \leq 2C' C^2 \epsilon / \delta
\text{(6)}
\]
by (4). For the second sum we have
\[
\sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} (e_t(u, v) - \frac{d_u d_v}{\text{vol}(G)})^2 = \sum_u \sum_{v \in G^*} \frac{1}{d_u d_v} \left( \sum_{z \in W^*} e_{t-1}(u, z) e(z, v) \right) \left( \frac{d_u d_v}{\text{vol}(G)} \right) = \sum_u \sum_{v \in G^*} \frac{1}{d_u d_v} \left( \sum_{z \in W^*} e_{t-1}(u, z) e(z, v) \right)^2 \leq \sum_u \sum_{v \in G^*} \frac{2}{d_u d_v} \left( \sum_{z \in W^*} e_{t-1}(u, z) e(z, v) \right)^2 + \left( \sum_{z \in W^*} e_{t-1}(u, z) e(z, v) \right) \frac{d_u d_v}{\text{vol}(G)}
\leq \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left( \sum_{z \in W^*} e_{t-1}(u, z) e(z, v) \right)^2
\leq \sum_u \sum_{v \notin X_u} \frac{1}{d_u d_v} \left( \sum_{z \in W^*} C d_u d_v e(z, v) \right)^2 \text{ by } U_{t-1}(C),
\leq \sum_u \sum_{v \notin X_u} \frac{C^2 d_u^2 (e(W^*, v))^2}{d_u d_v \text{vol}(G)^2}
\leq \sum_u \sum_{v \notin X_u} \frac{C^2 (C' + 1)^2 \delta d_u d_v}{\text{vol}(G)^2} \text{ by } (5)
\leq C^2 (C' + 1)^2 \delta.
\text{(7)}
For the second sum above, we have

\[
\sum_{u} \sum_{v \notin X_u} \frac{1}{d_u d_v} \left( \sum_{z \notin W^*} \frac{e_{i-1}(u, z) e(z, v)}{d_z} - \frac{d_u d_v}{\text{vol}(G)} \right)^2
\]

\[
= \sum_{u} \sum_{v \notin X_u} \frac{1}{d_u d_v} \left( \sum_{W_i \notin W^*} \sum_{z \in W_i} \frac{e_{i-1}(u, z) e(z, v)}{d_z} - \frac{d_u d_v}{\text{vol}(G)} \right)^2
\]

\[
\leq \sum_{u} \sum_{v \notin X_u} \frac{2}{d_u d_v} \left( \sum_{W_i \notin W^*} \sum_{z \in W_i} i \delta^{1/4} d_u d_z e(z, v) d_z \text{vol}(G) - \frac{d_u d_v}{\text{vol}(G)} \right)^2
\]

\[
+ \left( \sum_{W_i \notin W^*} \sum_{z \in W_i} \frac{\delta^{1/4} d_u d_z e(z, v) d_z}{\text{vol}(G)} \right)^2
\]

since \(|A - a| \leq b \Rightarrow (A - B)^2 \leq 2((a - B)^2 + b^2)\) and inequalities in (2).

For the second sum we have

\[
\sum_{u} \sum_{v \notin X_u} \frac{1}{d_u d_v} \left( \sum_{W_i \notin W^*} \sum_{z \in W_i} \frac{\delta^{1/4} d_u e(z, v)}{\text{vol}(G)} \right)^2
\]

\[
= \sqrt{\delta} \sum_{u} \sum_{v \notin X_u} \frac{1}{d_u d_v} \left( \sum_{W_i \notin W^*} \frac{d_u e(W_i, v)}{\text{vol}(G)} \right)^2
\]

\[
\leq \sqrt{\delta} \sum_{u} \sum_{v \notin X_u} \frac{1}{d_u d_v} \left( \sum_{W_i \notin W^*} \frac{d_u (1 + \sqrt{\delta}) d_v \text{vol}(W_i)}{\text{vol}(G)} \right)^2 \quad \text{by def. of } X_u
\]

\[
\leq \sqrt{\delta} \sum_{u} \sum_{v \notin X_u} \frac{1}{d_u d_v} \left( 1 + \sqrt{\delta} \frac{d_u d_v}{\text{vol}(G)} \right)^2
\]

\[
\leq \sqrt{\delta} + 3\delta
\]

(upper bounded by the sum over all \(u\) and \(v\)).
Finally, for the first sum we have

\[
\sum_u \sum_{v \in X_u} \frac{1}{d_u d_v} \left( \sum_{W_i \not\subseteq W^*} \frac{\sum z \in W_i \delta_1/4 d_u e(z, v)}{\text{vol}(G)} - \frac{d_u d_v}{\text{vol}(G)} \right)^2
\]

\[
= \sum_u \sum_{v \in X_u} \frac{1}{d_u d_v} \left( \sum_{W_i \not\subseteq W^*} \frac{\sum z \in W_i \delta_1/4 d_u e(W_i, v)}{\text{vol}(G)^2} - \frac{d_u d_v}{\text{vol}(G)} \right)^2
\]

\[
\leq \sum_u \sum_{v \in X_u} \frac{2}{d_u d_v} \left( \sum_{W_i \not\subseteq W^*} \frac{\sum z \in W_i \delta_1/4 d_u e(W_i)}{\text{vol}(G)^2} - \frac{d_u d_v}{\text{vol}(G)} \right)^2
\]

\[+ \left( \sum_{W_i \not\subseteq W^*} \frac{\sum z \in W_i \delta_1/4 d_u e(W_i) \sqrt{\delta}}{\text{vol}(G)^2} \right)^2 \]

by the def. of \( X_u \),

\[
\leq \sum_u \sum_{v \in X_u} \frac{2}{d_u d_v} \left( \sum_{W_i \not\subseteq W^*} \frac{\sum z \in W_i \delta_1/4 d_u e(W_i)}{\text{vol}(G)^2} - \frac{d_u d_v}{\text{vol}(G)} \right)^2
\]

\[+ 2 \left( \sum_{W_i \not\subseteq W^*} \frac{\sum z \in W_i \delta_1/4 d_u e(W_i)}{\text{vol}(G)^2} \right)^2 + \left( \sum_{W_i \not\subseteq W^*} \frac{\sum z \in W_i \delta_1/4 d_u e(W_i) \sqrt{\delta}}{\text{vol}(G)^2} \right)^2 \]

\[
\leq \sum_{u,v} \frac{1}{d_u d_v} \left( 4 \left( \frac{d_u d_v \delta_1/4}{\text{vol}(G)^2} \right)^2 + 4 \left( \frac{C^2 d_u \delta_1/4 d_u d_v \delta_2}{\text{vol}(G)^2} \right)^2 + 2 \left( \frac{C^2 d_u \delta_1/4 d_u \delta_2}{\text{vol}(G)} \right)^2 \right)
\]

by (3), def. of \( W^* \) and the fact that \( i < C' \),

\[
\leq 4\sqrt{\delta} + 4C'^4 \delta_3/2 + 2C'^2 \delta_3/2. \tag{9}
\]

Now, we have to put everything together.

First observe that

\[
\sum_{u,v \in V} \frac{1}{d_u d_v} (e_t(u, v) - \frac{d_u d_v}{\text{vol}(G)})^2
\]

\[
= \sum_{u,v \in V} \frac{1}{d_u d_v} e_t(u, v)^2 - 2 \sum_{u,v \in V} \frac{e_t(u, v)}{\text{vol}(G)} + \sum_{u,v \in V} \frac{d_u d_v}{\text{vol}(G)^2}
\]

\[
= \sum_{C_2,2t-circuit} w(C_{2t}) - \frac{e_t(V, V)}{\text{vol}(G)} + 1
\]

\[
= \sum_{C_2,2t-circuit} w(C_{2t}) - 1
\]
so that the preceding results, including inequalities (6), (7), (8) and (9), give

\[ | \sum_{C_{2t} \text{-circuit}} w(C_{2t}) - 1 | \]

\[ = \sum_{u,v \in V} \frac{1}{d_u d_v} \left( \epsilon(u,v) - \frac{d_u d_v}{\text{vol}(G)} \right)^2 \]

\[ \leq 2C^2C\epsilon/\delta + 2C^2(C' + 1)^2\delta + 4(\sqrt{\delta} + 3\delta) + 4(4\sqrt{\delta} + 4C^4\delta^{3/2} + 2C^2\delta^{3/2}) \]

\[ \leq 2C^2C\epsilon/\delta + 2C^2(C' + 1)^2\delta + 20\sqrt{\delta} + 12\delta + 16C^4\delta^{3/2} + 8C^2\delta^{3/2}. \]

This proves Theorem 2.

\[ \square \]

**Corollary 1** If \( G \) has minimum degree \( \alpha n \), then

\[ \text{DISC}(\epsilon) \Rightarrow \text{CIRCUIT}_{2t}(\eta) \]

where \( \eta \) depends only on \( \epsilon \) and \( \alpha \).

**Theorem 3** If \( G \) has minimum degree \( \alpha n \) for some constant \( \alpha \), then \( \text{CIRCUIT}_{2t}, \text{TRACE}_{2t}, \text{EIG}, \text{DISC}, \text{DISC}_2, \text{DISC}_t \) are all equivalent for \( t \geq 2 \).

### 7 Concluding remarks

We can summarize the main theorems in the following:

\[
\begin{align*}
\text{CIRCUIT}_{2t} & \iff \epsilon \\
\text{TRACE}_{2t} & \implies \epsilon^{1/2t} \implies \text{EIG} \implies \epsilon \implies \text{DISC} \\
& \implies \epsilon \implies \text{DISC}_2 \\
& \implies \epsilon \implies \text{DISC}_t \\
& \implies \epsilon \implies \text{DISC}_t \\
\end{align*}
\]

We should note that if for our degree sequence \( d \), we choose all \( d_i \) to be (approximately) equal, so that the \( G \in \mathcal{G}(d) \) are (approximately) regular, then our results specialize to the case of sparse random graphs considered in [19], except that here we get explicit functions of \( \epsilon \) (as opposed to the expressions with \( o(1) \) terms occurring in [19]). What are other properties which might be included in Theorem 1? Can condition \( U_{t-1} \) be replaced by a weaker condition to allow \( \text{DISC} \Rightarrow \text{CIRCUIT}_{2t} \) to be proved? We hope to return to this in the future.

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References


