An induced subgraph of the collaboration graph (with Erdos number at most 2).
Made by Fan Chung Graham and Lincoln Lu in 2002.
Announcements

About homework, write your own homework, allowing oral discussion with one fixed partner.
tree
Trees with at most 4 edges
G is a tree on n vertices.

G is connected with no cycle.

G is connected with n-1 edges.

G is formed by adding a leaf to a tree of n-1 vertices.

There is a unique path between any two vertices.
A binary tree
A k-level complete binary tree has ?? vertices.
A binary heap is a useful data structure if you have a collection of objects to which you are adding more objects and need to remove the object with the lowest value.
Chapter 4
Greedy Algorithms
4.5 Minimum Spanning Tree
Minimum Spanning Tree

**Minimum spanning tree.** Given a connected graph $G = (V, E)$ with real-valued edge weights $c_e$, an MST is a subset of the edges $T \subseteq E$ such that $T$ is a spanning tree whose sum of edge weights is minimized.

$$G = (V, E)$$

$$T, \sum_{e \in T} c_e = 50$$

**Cayley's Theorem.** There are $n^{n-2}$ spanning trees of $K_n$.

can't solve by brute force
Applications

MST is fundamental problem with diverse applications.

- **Network design.**
  - telephone, electrical, hydraulic, TV cable, computer, road

- **Approximation algorithms for NP-hard problems.**
  - traveling salesperson problem, Steiner tree

- **Indirect applications.**
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network

- **Cluster analysis.**
Kruskal's algorithm. Start with $T = \emptyset$. Consider edges in ascending order of cost. Insert edge $e$ in $T$ unless doing so would create a cycle.

Reverse-Delete algorithm. Start with $T = E$. Consider edges in descending order of cost. Delete edge $e$ from $T$ unless doing so would disconnect $T$.

Prim's algorithm. Start with some root node $s$ and greedily grow a tree $T$ from $s$ outward. At each step, add the cheapest edge $e$ to $T$ that has exactly one endpoint in $T$.

Remark. All three algorithms produce an MST.
Lexicographic Tiebreaking

To remove the assumption that all edge costs are distinct: perturb all edge costs by tiny amounts to break any ties.

Impact. Kruskal and Prim only interact with costs via pairwise comparisons. If perturbations are sufficiently small, MST with perturbed costs is MST with original costs. e.g., if all edge costs are integers, perturbing cost of edge \( e_i \) by \( i / n^2 \)

Implementation. Can handle arbitrarily small perturbations implicitly by breaking ties lexicographically, according to index.

```java
boolean less(i, j) {
    if (cost(e_i) < cost(e_j)) return true
    else if (cost(e_i) > cost(e_j)) return false
    else if (i < j) return true
    else return false
}
```
Cycles and Cuts

**Cycle.** Set of edges the form a-b, b-c, c-d, ..., y-z, z-a.

Cycle $C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1$

**Cutset.** A cut is a subset of nodes $S$. The corresponding cutset $D$ is the subset of edges with exactly one endpoint in $S$.

Cut $S = \{4, 5, 8\}$

Cutset $D = 5-6, 5-7, 3-4, 3-5, 7-8$
**Greedy Algorithms**

**Simplifying assumption.** All edge costs $c_e$ are distinct.

**Cut property.** Let $S$ be any subset of nodes, and let $e$ be the min cost edge with exactly one endpoint in $S$. Then the MST contains $e$.

**Cycle property.** Let $C$ be any cycle, and let $f$ be the max cost edge belonging to $C$. Then the MST does not contain $f$. 

![Diagram](image.png)
Prim’s Algorithm

\[ X = \{ \}, S = \{r\} \]
Repeat until \(S\) has \(n\) nodes:
- Pick the **lightest** edge \(e\) in the cut \((S, V - S)\)
- Add \(e\) to \(X\)
- Add \(v\), the end-point of \(e\) in \(V - S\) to \(S\)
**Data Structure: Heap**

**Heap Property:** If $x$ is the parent of $y$, then $\text{key}(x) \leq \text{key}(y)$

A heap is stored as a **balanced binary tree**

**Height** = $O(\log n)$, where $n = \# \text{ nodes}$
**Heap Property:** If $x$ is the parent of $y$, then $\text{key}(x) \leq \text{key}(y)$
Heap: Reporting the min

Heap Property: If \( x \) is the parent of \( y \), then \( \text{key}(x) \leq \text{key}(y) \)

Report the root node

Time = \( O(1) \)
Heap Property: If $x$ is the parent of $y$, then $\text{key}(x) \leq \text{key}(y)$

Add item $u$ to the end of the heap

If heap property is violated, swap $u$ with its parent
Heap: Add an item

Heap Property: If x is the parent of y, then key(x) <= key(y)

Add item u to the end of the heap
If heap property is violated, swap u with its parent
Heap: Add an item

Heap Property: If x is the parent of y, then key(x) <= key(y)

Add item u to the end of the heap
If heap property is violated, swap u with its parent
Heap: Add an item

**Heap Property:** If x is the parent of y, then key(x) \( \leq \) key(y)

Add item u to the end of the heap

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Heap Property: If x is the parent of y, then key(x) ≤ key(y)

Add item u to the end of the heap

If heap property is violated, swap u with its parent
Heap Property: If x is the parent of y, then key(x) <= key(y)

Add item u to the end of the heap
If heap property is violated, swap u with its parent

Time = O(log n)
Heap Property: If x is the parent of y, then $\text{key}(x) \leq \text{key}(y)$

Delete item $u$

Move v, the last item to $u$’s position
Heap Property: If x is the parent of y, then key(x) <= key(y)

If heap property is violated:

**Case 1.** $\text{key}[v] > \text{key}[	ext{child}[v]]$

**Case 2.** $\text{key}[v] < \text{key}[	ext{parent}[v]]$
Heap Property: If $x$ is the parent of $y$, then $\text{key}(x) \leq \text{key}(y)$

If heap property is violated:

**Case 1.** $\text{key}[v] > \text{key}[\text{child}[v]]$

Swap $v$ with its **lowest key** child
**Heap Property:** If \( x \) is the parent of \( y \), then \( \text{key}(x) \leq \text{key}(y) \)

If heap property is violated:

**Case 1.** \( \text{key}[v] > \text{key}[\text{child}[v]] \)

Swap \( v \) with its **lowest key** child

Continue until heap property holds

Time = \( O(\log n) \)
**Heap Property:** If \( x \) is the parent of \( y \), then \( \text{key}(x) \leq \text{key}(y) \)

If heap property is violated:

**Case 2.** \( \text{key}[v] < \text{key}[\text{parent}[v]] \)

Swap \( v \) with its **parent**

Continue till heap property holds

Time = \( O(\log n) \)

\( \text{key}(x) \)
Heap Property: If x is the parent of y, then key(x) \( \leq \) key(y)

If heap property is violated:

**Case 2.** \( \text{key}[v] < \text{key}[	ext{parent}[v]] \)

Swap v with its parent

Continue till heap property holds

Time = \( O(\log n) \)
**Heap Property:** If x is the parent of y, then key(x) <= key(y)

If heap property is violated:

**Case 2.** key[v] < key[parent[v]]

Swap v with its **parent**

Continue till heap property holds
Heap: Delete an item

**Heap Property:** If x is the parent of y, then key(x) <= key(y)

If heap property is violated:

**Case 2.** key[v] < key[parent[v]]

Swap v with its parent

Continue till heap property holds
Summary: Heap

Heap Property: If x is the parent of y, then key(x) <= key(y)

Operations:
- Add an element: O(log n)
- Delete an element: O(log n)
- Report min: O(1)
Prim’s Algorithm

X = {}, S = {r}
Repeat until S has n nodes:
   1. Pick the **lightest** edge e in the cut (S, V - S)
   2. Delete all edges b/w v and S from heap
   3. Add all edges b/w v and V - S - {v}

Use a **heap** to store edges between S and V - S
At each step:
   1. Pick lightest edge with a report-min
   2. Delete all edges b/w v and S from heap
   3. Add all edges b/w v and V - S - {v}

Black edges = in heap
## Prim’s Algorithm

X = { }, S = {r}
Repeat until S has n nodes:
1. Pick the **lightest** edge e in the cut (S, V - S)
2. Delete all edges b/w v and S from heap
3. Add all edges b/w v and V - S - {v}

#edge additions and deletions = O(m) (Why?)
#report mins = O(n)

Use a **heap** to store edges between S and V - S
At each step:
1. Pick lightest edge with a report-min
2. Delete all edges b/w v and S from heap
3. Add all edges b/w v and V - S - {v}

Black edges = in heap
Prim’s Algorithm

\[ X = \{ \}, S = \{ r \} \]
Repeat until \( S \) has \( n \) nodes:

1. Pick the **lightest** edge \( e \) in the cut \((S, V - S)\)
2. Add \( e \) to \( X \)
3. Add \( v \), the end-point of \( e \) in \( V - S \) to \( S \)

Use a **heap** to store edges b/w \( S \) and \( V - S \)
At each step:

1. Pick lightest edge with a report-min
2. Delete all edges b/w \( v \) and \( S \) from heap
3. Add all edges b/w \( v \) and \( V - S - \{ v \} \)

\#edge additions and deletions = \( O(m) \)
\#report mins = \( O(n) \)

**Heap Ops:**
- Add: \( O(\log n) \)
- Delete: \( O(\log n) \)
- Report min: \( O(1) \)
Prim’s Algorithm

X = { }, S = {r}
Repeat until S has n nodes:
   Pick the **lightest** edge e in the cut (S, V - S)
   Add e to X
   Add v, the end-point of e in V - S to S

Use a **heap** to store edges b/w S and V - S
At each step:
   1. Pick lightest edge with a report-min
   2. Delete all edges b/w v and S from heap
   3. Add all edges b/w v and V - S - {v}

#edge additions and deletions = O(m)
#report mins = O(n)
Total running time = O(m log n)
**Kruskal's Algorithm**

**Kruskal's algorithm.** [Kruskal, 1956]

- Consider edges in ascending order of weight.
- **Case 1:** If adding \( e \) to \( T \) creates a cycle, discard \( e \) according to cycle property.
- **Case 2:** Otherwise, insert \( e = (u, v) \) into \( T \) according to cut property where \( S \) = set of nodes in \( u \)'s connected component.

![Case 1](image1)

![Case 2](image2)
Implementation: Kruskal’s Algorithm

Implementation. Use the union-find data structure.
- Build set $T$ of edges in the MST.
- Maintain set for each connected component.
- $O(m \log n)$ for sorting and $O(m \alpha (m, n))$ for union-find.

$m \leq n^2 \Rightarrow \log m$ is $O(\log n)$ essentially a constant

```
Kruskal(G, c) {
    Sort edges weights so that $c_1 \leq c_2 \leq \ldots \leq c_m$.
    $T \leftarrow \emptyset$

    foreach $(u \in V)$ make a set containing singleton $u$

    for $i = 1$ to $m$
        $(u, v) = e_i$
        if $(u$ and $v$ are in different sets) {
            $T \leftarrow T \cup \{e_i\}$
            merge the sets containing $u$ and $v$
        }
    return $T$
}
```
Union-Find Implementation

\textbf{Find}(x) - follow pointers from }x\text{ up to root

\textbf{Union}(x, y) - make }x\text{ a child of }y\text{ and return }y
Path Compression

\[ \text{Find}(c) \]

Sunday, April 13, 14
Summary: Kruskal’s Algorithm
Running Time

\[ X = \{ \} \]

For each edge \( e \) in **increasing order** of weight:
   
   If the end-points of \( e \) lie in different components in \( X \),
   
   Add \( e \) to \( X \)

**Sort** the edges = \( O(m \log m) = O(m \log n) \)

**Add e** to \( X \) = Union Operation = \( O(1) + \text{Time(Find)} \)

**Check** if end-points of \( e \) lie in different components = Find Operation

Total time = Sort + \( O(n) \) Unions + \( O(m) \) Finds = \( O(m \log n) \)

With sorted edges, time = \( O(n) \) Unions + \( O(m) \) Finds = \( O(m \log^* n) \)
The function $\text{lg}^* n$

$\text{lg}^* n$ = the number of times we have to take the $\log_2$ of $n$ repeatedly to reach 1

- $\text{lg}^* 2 = 1$
- $\text{lg}^* 3 = \text{lg}^* 4 = \text{lg}^* 2^2 = 2$
- $\text{lg}^* 16 = \text{lg}^* 2^{2^2} = 3$
- $\text{lg}^* 65536 = \text{lg}^* 2^{2^{2^2}} = 4$

$\Rightarrow \text{lg}^* n \leq 5$ for all practical values of $n$
Theorem (**Tarjan**): If

\[ S = \text{a sequence of } O(n) \text{ Unions and Find-Sets} \]

The worst-case time for \( S \) with

- Weighted Unions, and
- Path Compressions

is \( O(n \lg^* n) \)

\[ \Rightarrow \text{The average time is } O(\lg^* n) \text{ per operation} \]
Ackerman's function

Define the function $A_i(x)$ inductively by

\[ A_0(x) = x+1 \]
\[ A_{i+1}(x) = A_i(A_i(A_i\ldots(x))), \text{ where } A_i \text{ is applied } x+1 \text{ times.} \]

\[ A_1(x) = 2x+1 \]
\[ A_2(x) = A_1(A_1(A_1\ldots(x))) > 2^{x+1} \]
\[ A_3(x) = A_2(A_2(A_2\ldots(x))) > 2^{2^{2^{2\ldots(x)}}} \]

The Inverse Ackerman function is

\[ \alpha(n) = \min\{k: A_k(1) > n \} \]
Theorem (Tarjan): Let 
\( S = \) sequence of \( \Omega(n) \) Unions and Find-Sets

The worst-case time for \( S \) with

- Weighted Unions, and
- Path Compressions

is \( O(n\alpha(n)) \)

\[ \Rightarrow \] The average time is \( O(\alpha(n)) \) steps per operation
4.7 Clustering

Outbreak of cholera deaths in London in 1850s.
Reference: Nina Mishra, HP Labs
Single Linkage Clustering

**Problem**: Given a set of points, build a hierarchical clustering.

**Procedure**:
- Initialize: each node is a cluster
- Until we have one cluster:
  - Pick two **closest** clusters $C, C^*$
  - **Merge** $S = C \cup C^*$

**Distance between two clusters**:
$$d(C, C^*) = \min_{x \in C, y \in C^*} d(x, y)$$

Can you recognize this algorithm?
Clustering

Clustering. Given a set $U$ of $n$ objects labeled $p_1, \ldots, p_n$, classify into coherent groups.

Distance function. Numeric value specifying "closeness" of two objects.

Fundamental problem. Divide into clusters so that points in different clusters are far apart.

- Routing in mobile ad hoc networks.
- Identify patterns in gene expression.
- Document categorization for web search.
- Similarity searching in medical image databases
- Skycat: cluster $10^9$ sky objects into stars, quasars, galaxies.
Clustering of Maximum Spacing

**k-clustering.** Divide objects into k non-empty groups.

**Distance function.** Assume it satisfies several natural properties.
- \( d(p_i, p_j) = 0 \iff p_i = p_j \) (identity of indiscernibles)
- \( d(p_i, p_j) \geq 0 \) (nonnegativity)
- \( d(p_i, p_j) = d(p_j, p_i) \) (symmetry)

**Spacing.** Min distance between any pair of points in different clusters.

**Clustering of maximum spacing.** Given an integer k, find a k-clustering of maximum spacing.

---

[Diagram showing a set of points clustered into groups with a minimum distance between clusters marked as 'spacing'.]
Greedy Clustering Algorithm

Single-link k-clustering algorithm.
- Form a graph on the vertex set $U$, corresponding to $n$ clusters.
- Find the closest pair of objects such that each object is in a different cluster, and add an edge between them.
- Repeat $n-k$ times until there are exactly $k$ clusters.

Key observation. This procedure is precisely Kruskal’s algorithm (except we stop when there are $k$ connected components).

Remark. Equivalent to finding an MST and deleting the $k-1$ most expensive edges.
**Greedy Clustering Algorithm: Analysis**

**Theorem.** Let $C^*$ denote the clustering $C^*_1, ..., C^*_k$ formed by deleting the $k-1$ most expensive edges of a MST. $C^*$ is a $k$-clustering of max spacing.

**Pf.** Let $C$ denote some other clustering $C_1, ..., C_k$.

- The spacing of $C^*$ is the length $d^*$ of the $(k-1)^{st}$ most expensive edge.
- Let $p_i, p_j$ be in the same cluster in $C^*$, say $C^*_r$, but different clusters in $C$, say $C_s$ and $C_t$.
- Some edge $(p, q)$ on $p_i$-$p_j$ path in $C^*_r$ spans two different clusters in $C$.
- All edges on $p_i$-$p_j$ path have length $\leq d^*$ since Kruskal chose them.
- Spacing of $C$ is $\leq d^*$ since $p$ and $q$ are in different clusters.

![Diagram showing clustering](attachment:image.png)
Extra Slides
MST Algorithms: Theory

Deterministic comparison based algorithms.

- $O(m \log n) \quad$ [Jarník, Prim, Dijkstra, Kruskal, Boruvka]
- $O(m \log \log n). \quad$ [Cheriton-Tarjan 1976, Yao 1975]
- $O(m \beta(m, n)). \quad$ [Fredman-Tarjan 1987]
- $O(m \log \beta(m, n)). \quad$ [Gabow-Galil-Spencer-Tarjan 1986]
- $O(m \alpha (m, n)). \quad$ [Chazelle 2000]

Holy grail. $O(m)$.

Notable.

- $O(m)$ randomized. \quad [Karger-Klein-Tarjan 1995]
- $O(m)$ verification. \quad [Dixon-Rauch-Tarjan 1992]

Euclidean.

- 2-d: $O(n \log n)$. \quad compute MST of edges in Delaunay
- k-d: $O(k n^2)$. \quad dense Prim