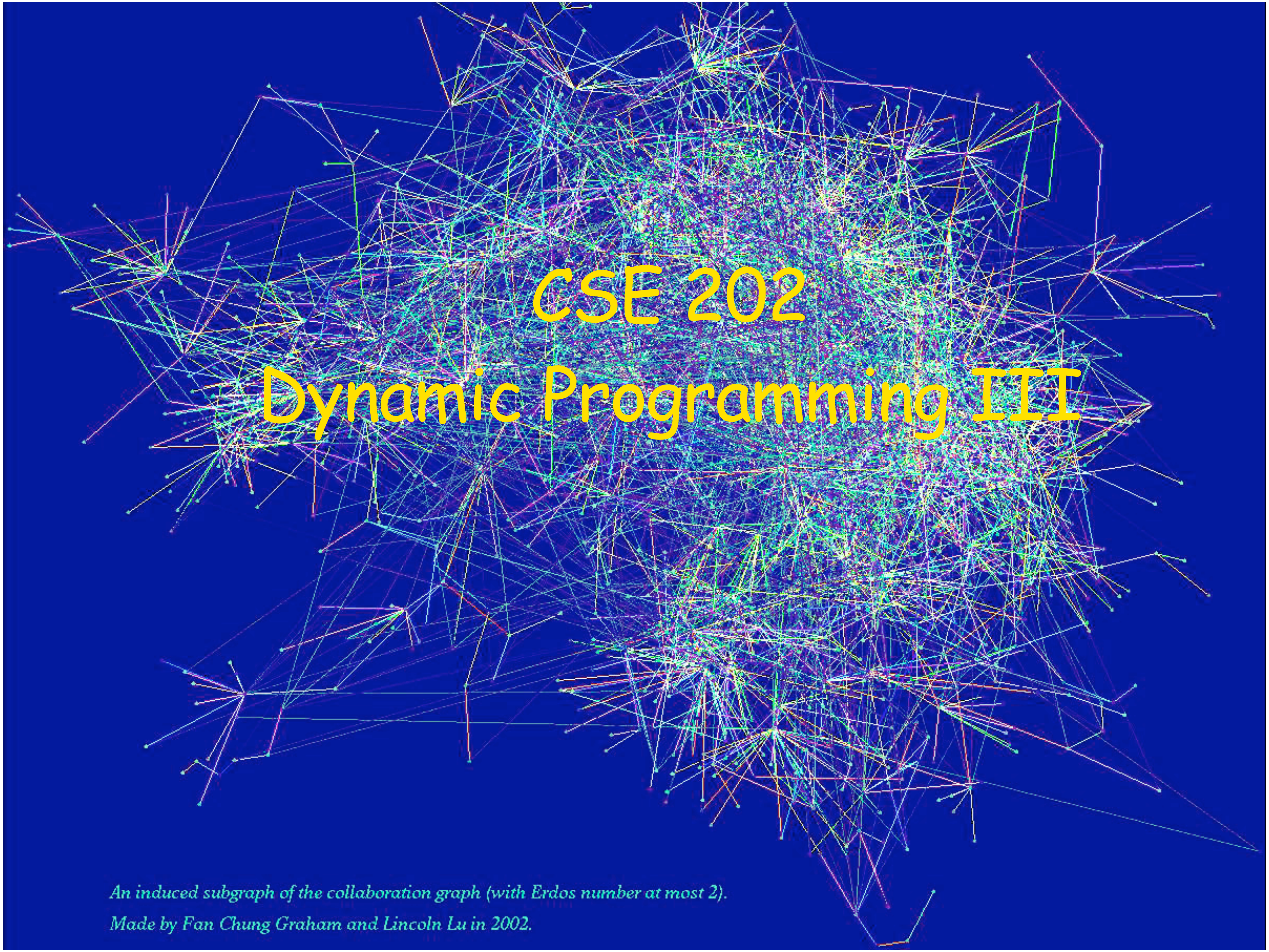
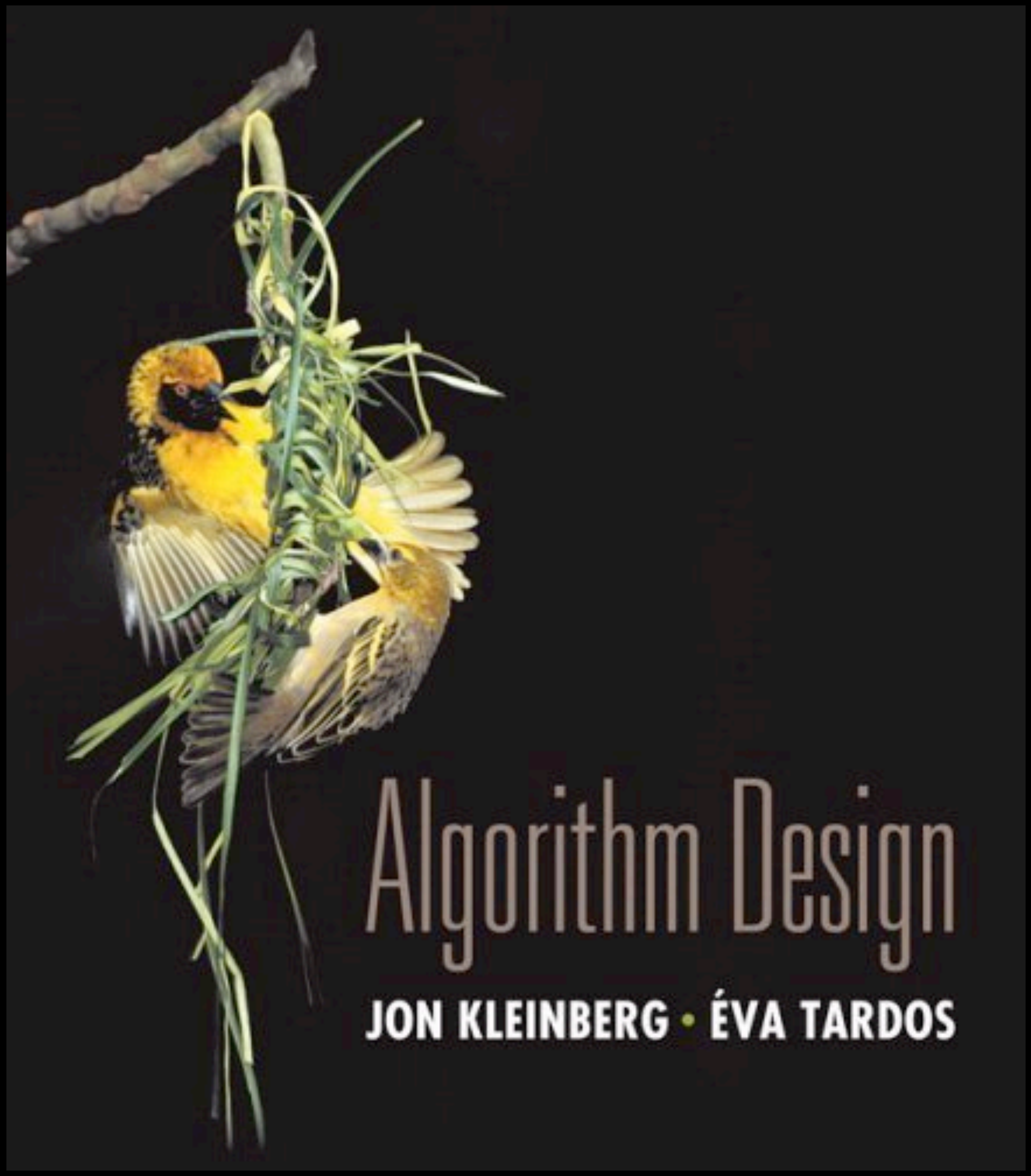
CSE 202
Dynamic Programming III



An induced subgraph of the collaboration graph (with Erdos number at most 2).
Made by Fan Chung Graham and Lincoln Lu in 2002.



Chapter 6

Dynamic Programming

Slides by Kevin Wayne.

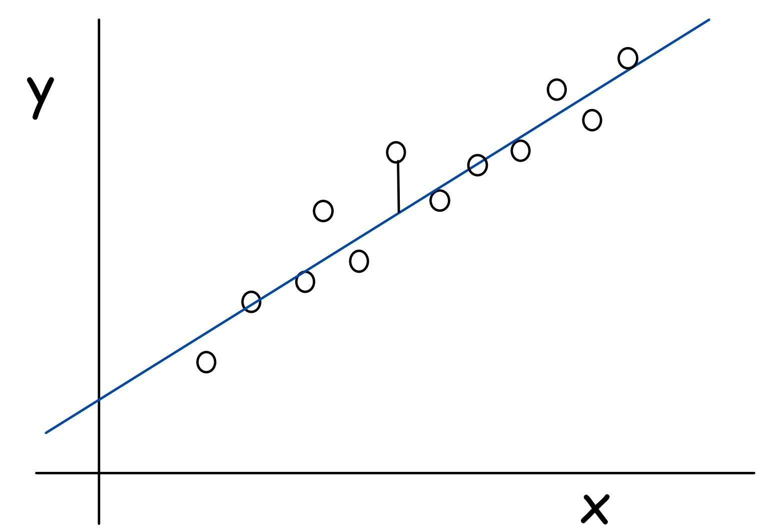
6.1 Weighted Interval Scheduling

6.3 Segmented Least Squares

Segmented Least Squares

Least squares.
- Foundational problem in statistic and numerical analysis.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
- Find a line y = ax + b that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$



Solution. Calculus $\Rightarrow$ min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented Least Squares

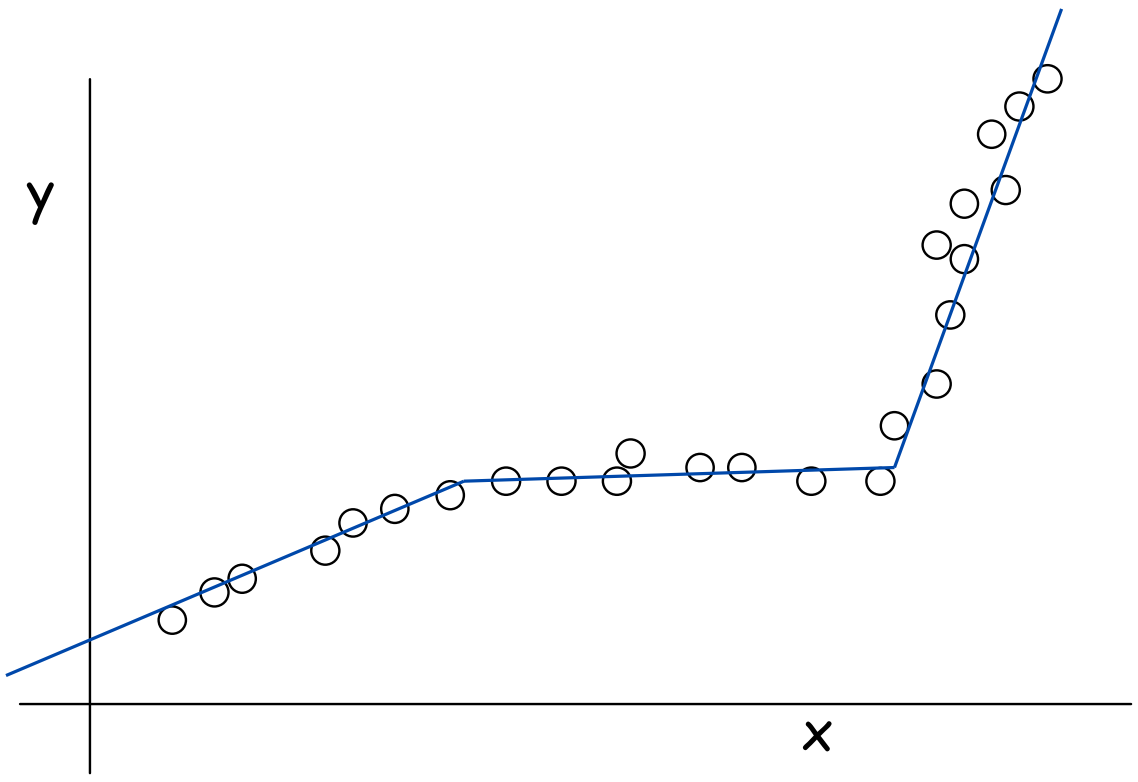
Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with
- $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes $f(x)$.

Q. What's a reasonable choice for f(x) to balance accuracy and parsimony?

↑ goodness of fit (accuracy)

↑ number of lines (parsimony)



Segmented Least Squares

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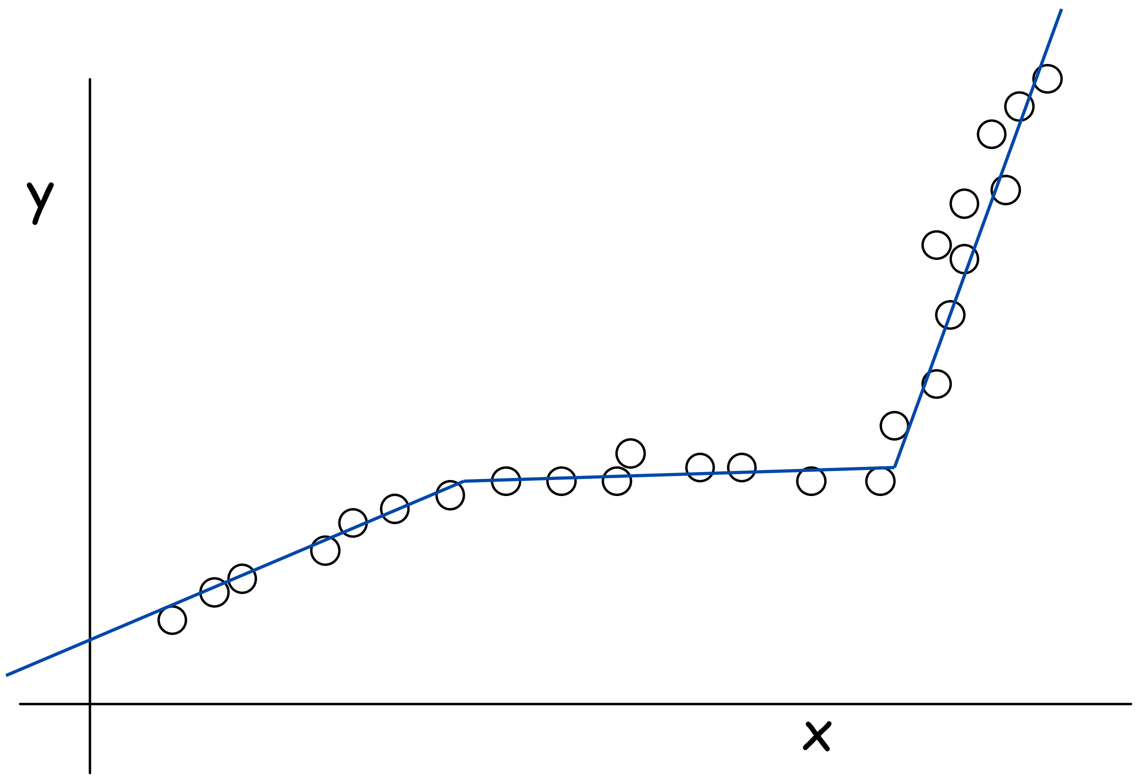
- $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes:
  - the sum of the sums of the squared errors E in each segment
  - the number of lines L
- Tradeoff function: E + c L, for some constant c > 0.



Dynamic Programming: Multiway Choice

Notation.

- $OPT(j)$ = minimum cost for points $p_1, p_{i+1}, \ldots, p_j$.
- $e(i, j)$ = minimum sum of squares for points $p_i, p_{i+1}, \ldots, p_j$.

To compute OPT(j):

- Last segment uses points $p_i, p_{i+1}, \ldots, p_j$ for some i.
- Cost = $e(i, j) + c + OPT(i-1)$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min\limits_{1 \le i \le j} \{ \; e(i,j) + c + OPT(i-1) \} & \text{otherwise} \end{cases}$$

Segmented Least Squares: Algorithm

```
INPUT: n, p1,…,pN , c

Segmented-Least-Squares() {
   M[0] = 0
   for j = 1 to n
      for i = 1 to j
         compute the least square error eij for
         the segment pi,…, pj

   for j = 1 to n
      M[j] = min 1 ≤ i ≤ j (eij + c + M[i-1])

   return M[n]
}
```

Running time. $O(n^3)$. ← can be improved to $O(n^2)$ by pre-computing various statistics

- Bottleneck = computing $e(i, j)$ for $O(n^2)$ pairs, $O(n)$ per pair using previous formula.

6.4 Knapsack Problem

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack."
- Item i weighs w_i > 0 kilograms and has value v_i > 0.
- Knapsack has capacity of W kilograms.
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
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Greedy: repeatedly add item with maximum ratio v_i / w_i.

Ex: { 5, 2, 1 } achieves only value = 35 $\Rightarrow$ greedy not optimal.

Dynamic Programming: False Start

Def. OPT(i) = max profit subset of items 1, …, i.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 }
- Case 2: OPT selects item i.
  - accepting item i does not immediately imply that we will have to reject other items
  - without knowing what other items were selected before i, we don't even know if we have enough room for i

Conclusion. Need more sub-problems!

Dynamic Programming: Adding a New Variable

Def. OPT(i, w) = max profit subset of items 1, …, i with weight limit w.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 } using weight limit w
- Case 2: OPT selects item i.
  - new weight limit = $w - w_i$
  - OPT selects best of { 1, 2, …, i–1 } using this new weight limit

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max\{\, OPT(i-1, w),\ \ v_i + OPT(i-1, w - w_i) \,\} & \text{otherwise} \end{cases}$$

Knapsack Problem: Bottom-Up

Knapsack. Fill up an n-by-W array.

```
Input: n, w1,…,wN, v1,…,vN

for w = 0 to W
   M[0, w] = 0

for i = 1 to n
   for w = 1 to W
      if (wi > w)
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max {M[i-1, w], vi + M[i-1, w-wi ]}

return M[n, W]
```

Knapsack Algorithm

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
φ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
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{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

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Knapsack Problem: Running Time

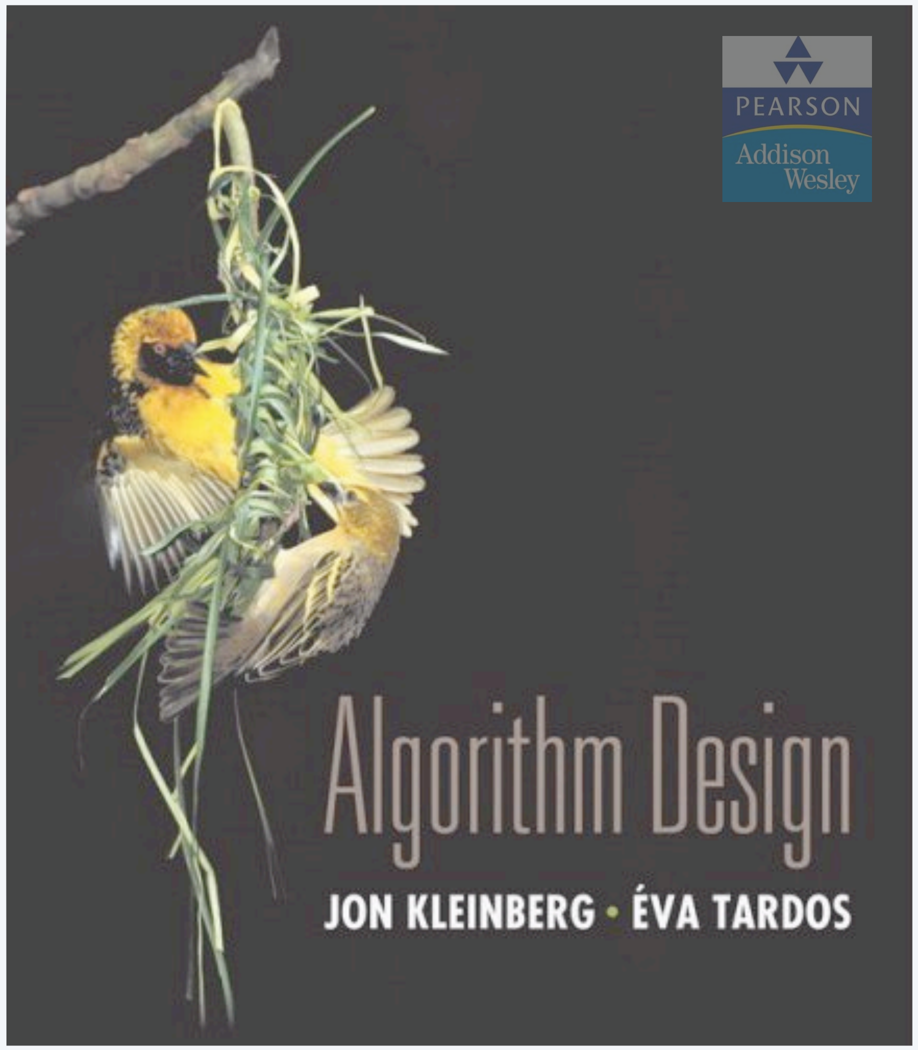
Running time. $\Theta(n\ W)$.

- Not polynomial in input size!
- "Pseudo-polynomial."
- Decision version of Knapsack is NP-complete. [Chapter 8]

Knapsack approximation algorithm. There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% of optimum. [Section 11.8]

6.5 RNA Secondary Structure

6.6 Sequence Alignment



SECTION 6.6

6. DYNAMIC PROGRAMMING II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman-Ford algorithm*
- *distance vector protocols*
- *negative cycles in a digraph*

String similarity

Q. How similar are two strings?

Ex. ocurrance and occurrence.

o	c	u	r	r	a	n	c	e	–
o	c	c	u	r	r	e	n	c	e

6 mismatches, 1 gap

o	c	–	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap

o	c	–	u	r	r	–	a	n	c	e
o	c	c	u	r	r	e	–	n	c	e

0 mismatches, 3 gaps

Edit distance

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]

- Gap penalty δ; mismatch penalty α_{pq}.
- Cost = sum of gap and mismatch penalties.

C	T	–	G	A	C	C	T	A	C	G
C	T	G	G	A	C	G	A	A	C	G

cost = $\delta + \alpha_{CG} + \alpha_{TA}$

Applications. Unix diff, speech recognition, computational biology, ...

Sequence alignment

Goal. Given two strings $x_1\, x_2 \ldots x_m$ and $y_1\, y_2 \ldots y_n$ find min cost alignment.

Def. An **alignment** M is a set of ordered pairs $x_i - y_j$ such that each item occurs in at most one pair and no crossings.

$x_i - y_j$ and $x_{i'} - y_{j'}$ cross if $i < i'$, but $j > j'$

Def. The **cost** of an alignment M is:

$$\text{cost}(M) = \underbrace{\sum_{(x_i, y_j) \in M} \alpha_{x_i y_j}}_{\text{mismatch}} + \underbrace{\sum_{i\,:\,x_i \text{ unmatched}} \delta + \sum_{j\,:\,y_j \text{ unmatched}} \delta}_{\text{gap}}$$

x_1	x_2	x_3	x_4	x_5		x_6
C	T	A	C	C	–	G
–	T	A	C	A	T	G
	y_1	y_2	y_3	y_4	y_5	y_6

an alignment of CTACCG and TACATG:

$$M = \{\, x_2 – y_1,\ x_3 – y_2,\ x_4 – y_3,\ x_5 – y_4,\ x_6 – y_6 \,\}$$

Sequence alignment: problem structure

Def. $OPT(i, j)$ = min cost of aligning prefix strings $x_1\, x_2 \ldots x_i$ and $y_1\, y_2 \ldots y_j$.

Case 1. OPT matches $x_i - y_j$.
Pay mismatch for $x_i - y_j$ + min cost of aligning $x_1\, x_2 \ldots x_{i-1}$ and $y_1\, y_2 \ldots y_{j-1}$.

Case 2a. OPT leaves x_i unmatched.
Pay gap for x_i + min cost of aligning $x_1\, x_2 \ldots x_{i-1}$ and $y_1\, y_2 \ldots y_j$.

optimal substructure property
(proof via exchange argument)

Case 2b. OPT leaves y_j unmatched.
Pay gap for y_j + min cost of aligning $x_1\, x_2 \ldots x_i$ and $y_1\, y_2 \ldots y_{j-1}$.

$$OPT(i, j) = \begin{cases} j\delta & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + OPT(i-1, j-1) \\ \delta + OPT(i-1, j) \\ \delta + OPT(i, j-1) \end{cases} & \text{otherwise} \\ i\delta & \text{if } j = 0 \end{cases}$$

Sequence alignment: algorithm

SEQUENCE-ALIGNMENT $(m, n, x_1, \ldots, x_m, y_1, \ldots, y_n, \delta, \alpha)$

FOR $i = 0$ TO m

  $M[i, 0] \leftarrow i\,\delta.$

FOR $j = 0$ TO n

  $M[0, j] \leftarrow j\,\delta.$

FOR $\mathrm{i} = 1$ TO m

  FOR $\mathrm{j} = 1$ TO n

$$M[i, j] \leftarrow \min \{\, \alpha[x_i, y_j] + M[i-1, j-1],\\ \delta + M[i-1, j],\\ \delta + M[i, j-1]).$$

RETURN $M[m, n]$.

Sequence alignment: analysis

Theorem. The dynamic programming algorithm computes the edit distance (and optimal alignment) of two strings of length m and n in $\Theta(mn)$ time and $\Theta(mn)$ space.

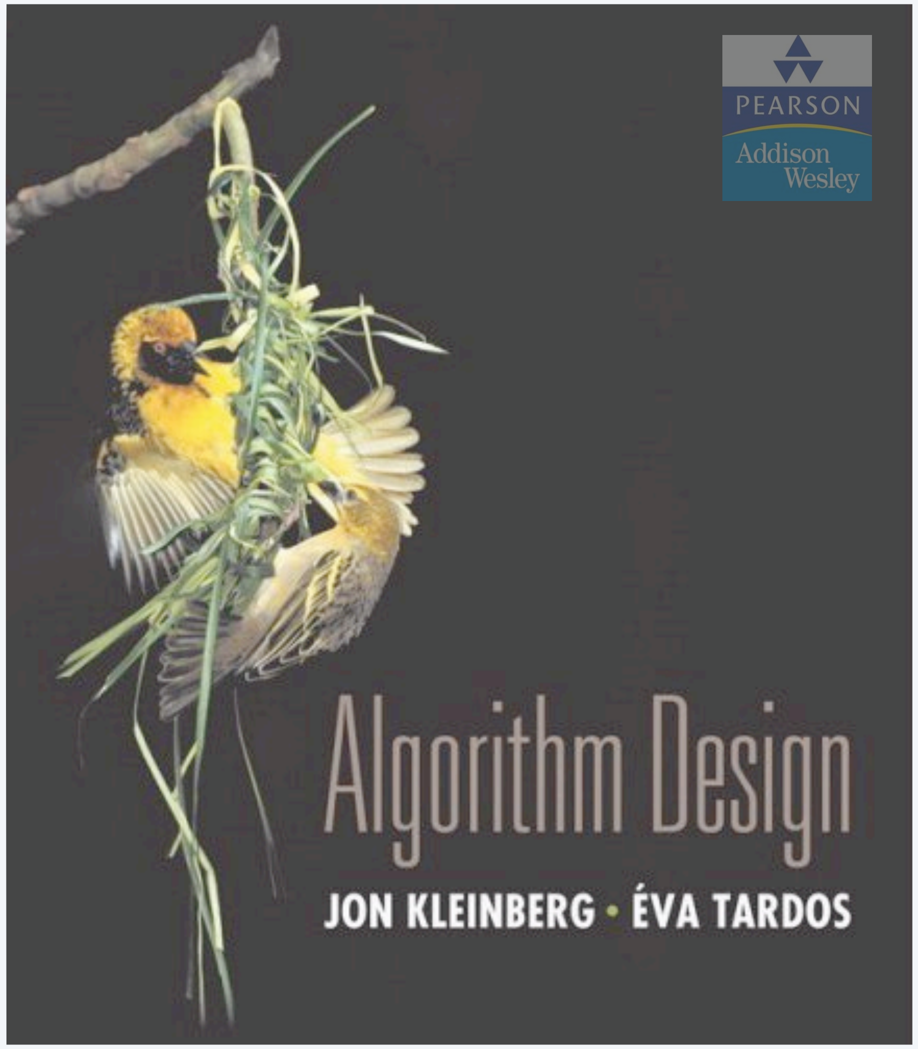
Pf.

- Algorithm computes edit distance.
- Can trace back to extract optimal alignment itself. ▪

Q. Can we avoid using quadratic space?

A. Easy to compute optimal value in $O(mn)$ time and $O(m + n)$ space.

- Compute $\text{OPT}(i, \bullet)$ from $\text{OPT}(i-1, \bullet)$.
- But, no longer easy to recover optimal alignment itself.



SECTION 6.7

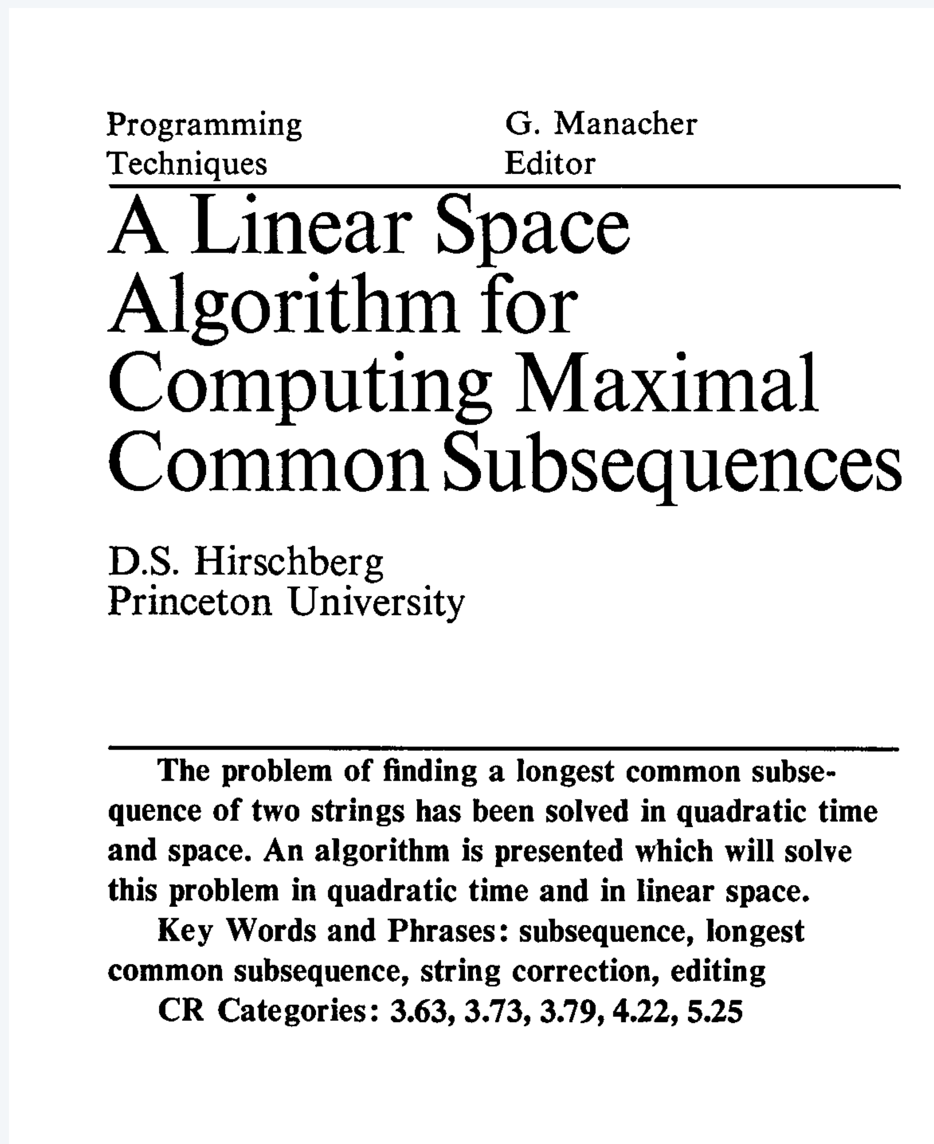
6. DYNAMIC PROGRAMMING II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman-Ford algorithm*
- *distance vector protocols*
- *negative cycles in a digraph*

Sequence alignment in linear space

Theorem. There exist an algorithm to find an optimal alignment in $O(mn)$ time and $O(m+n)$ space.

- Clever combination of divide-and-conquer and dynamic programming.
- Inspired by idea of Savitch from complexity theory.



Programming Techniques
G. Manacher Editor

A Linear Space Algorithm for Computing Maximal Common Subsequences

D.S. Hirschberg
Princeton University

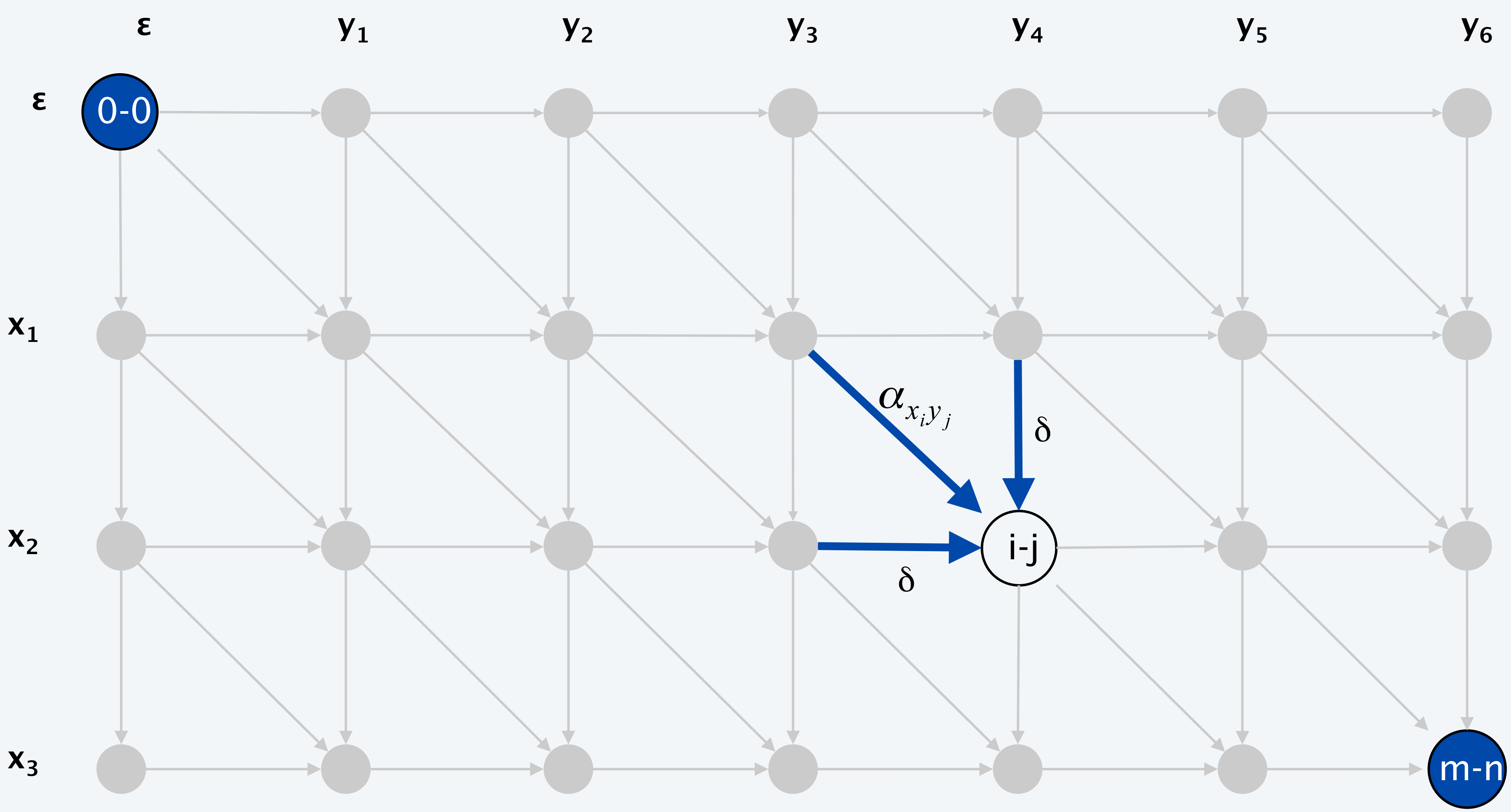
The problem of finding a longest common subsequence of two strings has been solved in quadratic time and space. An algorithm is presented which will solve this problem in quadratic time and in linear space.

Key Words and Phrases: subsequence, longest common subsequence, string correction, editing
CR Categories: 3.63, 3.73, 3.79, 4.22, 5.25

Hirschberg's algorithm

Edit distance graph.

- Let $f(i, j)$ be shortest path from $(0,0)$ to (i, j).
- Lemma: $f(i, j) = OPT(i, j)$ for all i and j.



Hirschberg's algorithm

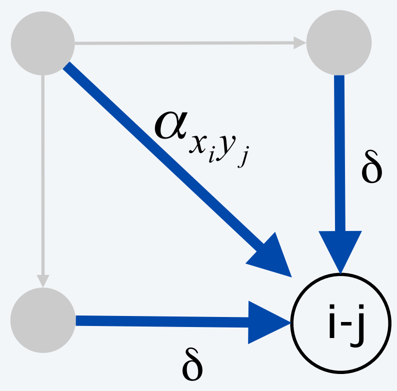
Edit distance graph.

- Let $f(i, j)$ be shortest path from $(0,0)$ to (i, j).
- Lemma: $f(i, j) = OPT(i, j)$ for all i and j.

Pf of Lemma. [by strong induction on $i + j$]

- Base case: $f(0, 0) = OPT(0, 0) = 0$.
- Inductive hypothesis: assume true for all (i', j') with $i' + j' < i + j$.
- Last edge on shortest path to (i, j) is from $(i - 1, j - 1)$, $(i - 1, j)$, or $(i, j - 1)$.
- Thus,

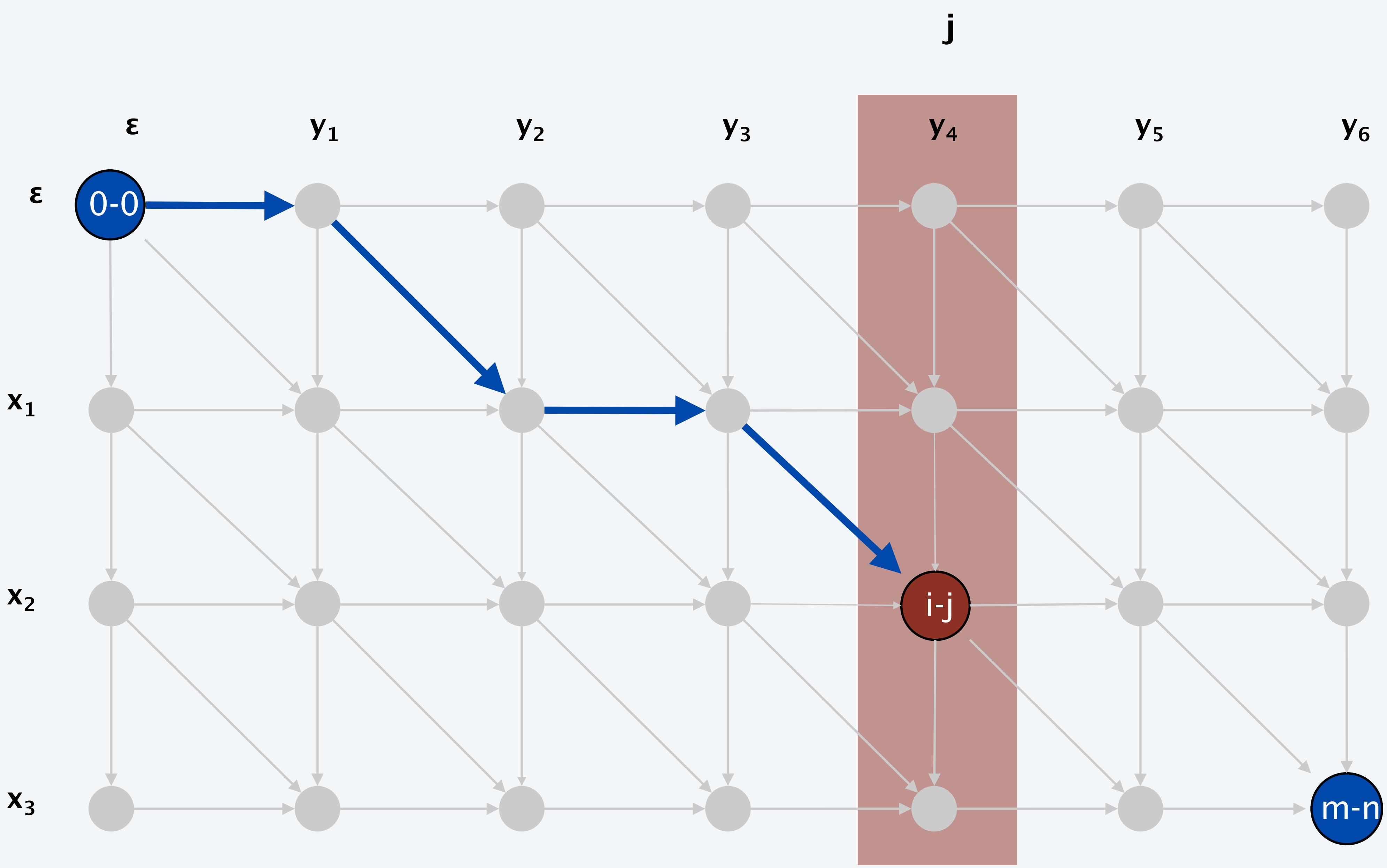
$$\begin{aligned} f(i,j) &= \min\{\alpha_{x_i y_j} + f(i-1,j-1),\ \delta + f(i-1,j),\ \delta + f(i,j-1)\} \\ &= \min\{\alpha_{x_i y_j} + OPT(i-1,j-1),\ \delta + OPT(i-1,j),\ \delta + OPT(i,j-1)\} \\ &= OPT(i,j)\ \blacksquare \end{aligned}$$



Hirschberg's algorithm

Edit distance graph.

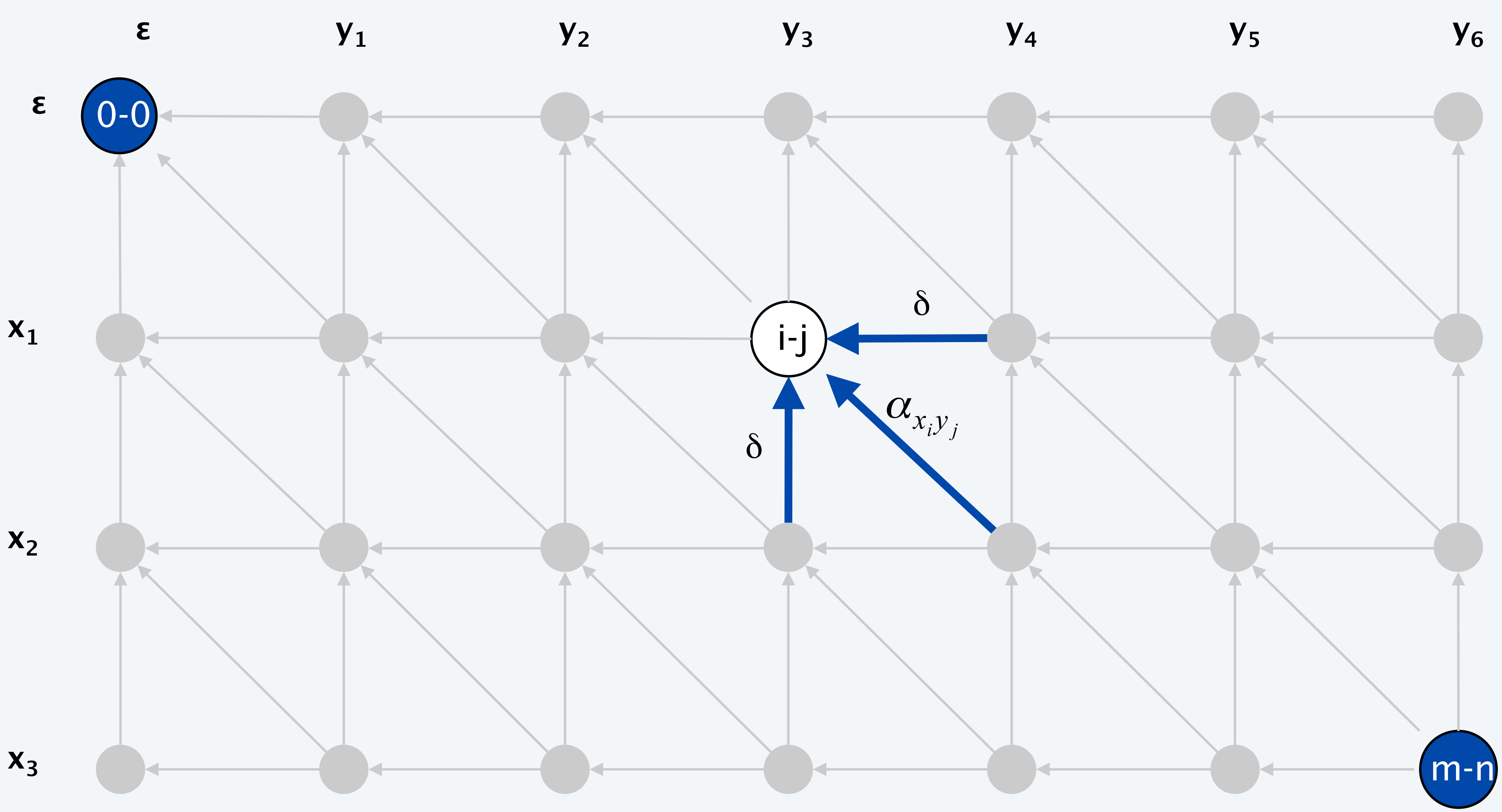
- Let $f(i, j)$ be shortest path from $(0,0)$ to (i, j).
- Lemma: $f(i, j) = OPT(i, j)$ for all i and j.
- Can compute $f(\bullet, j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



Hirschberg's algorithm

Edit distance graph.

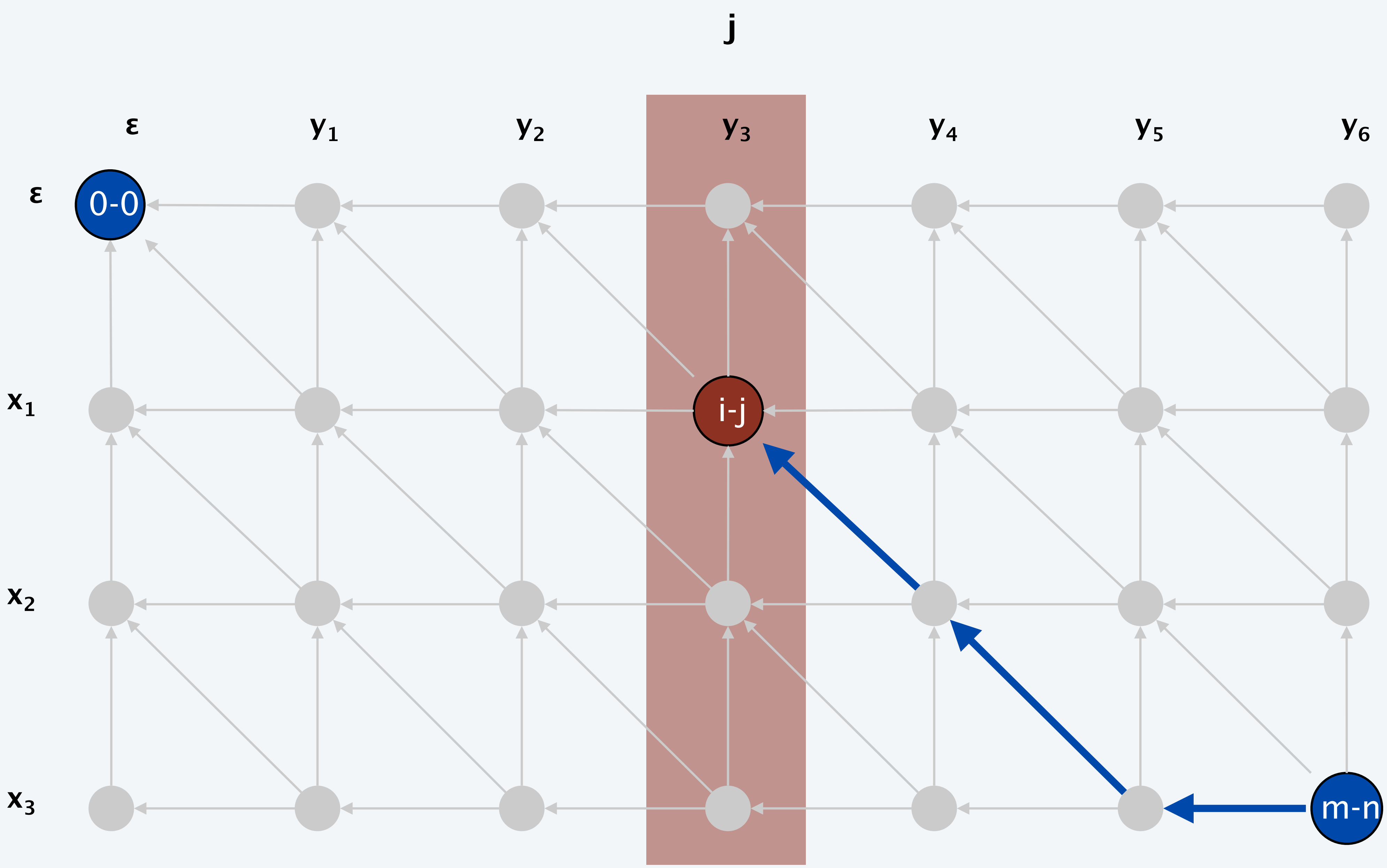
- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute by reversing the edge orientations and inverting the roles of $(0, 0)$ and (m, n).



Hirschberg's algorithm

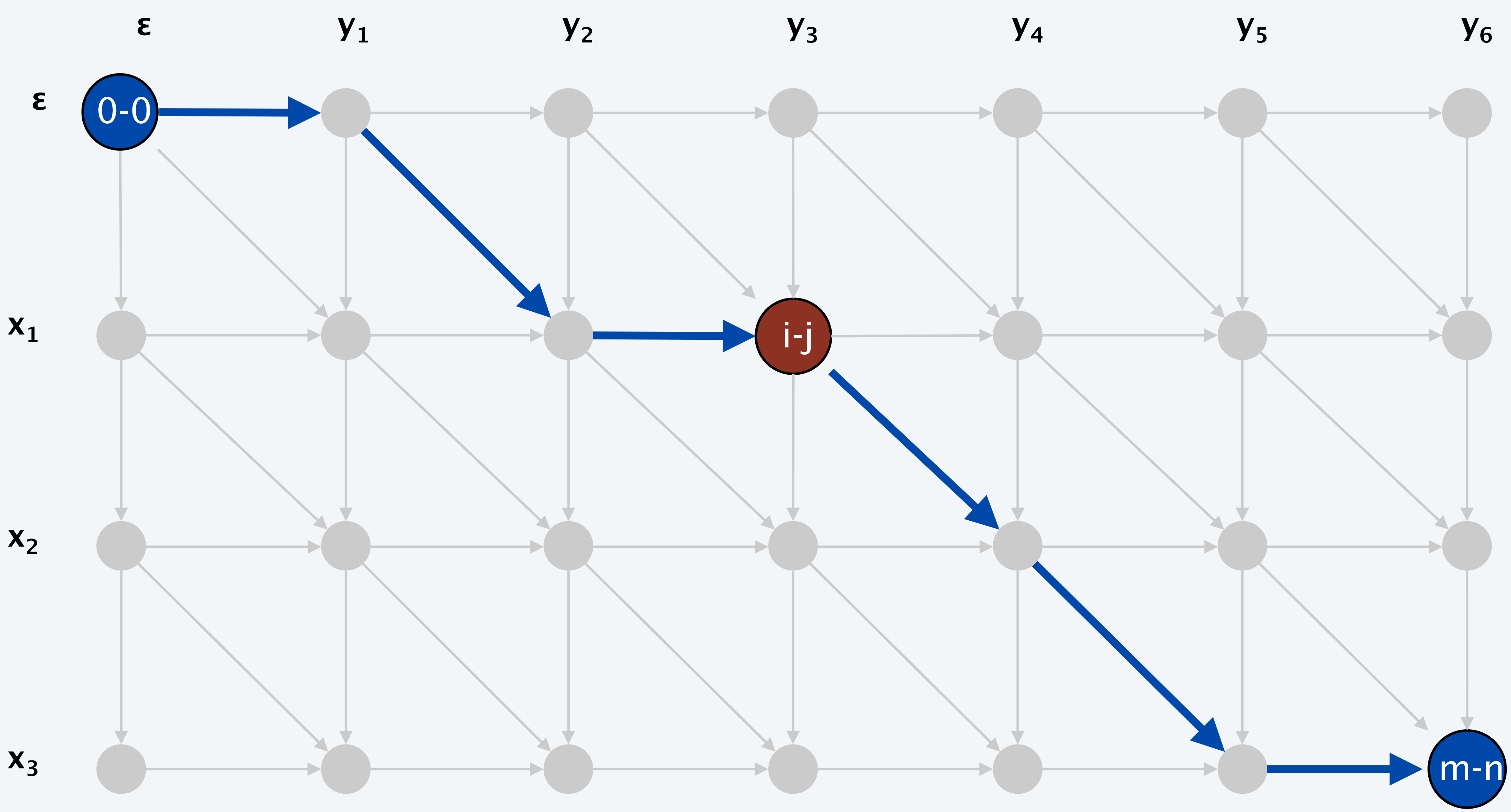
Edit distance graph.

- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute $g(\bullet,\ j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



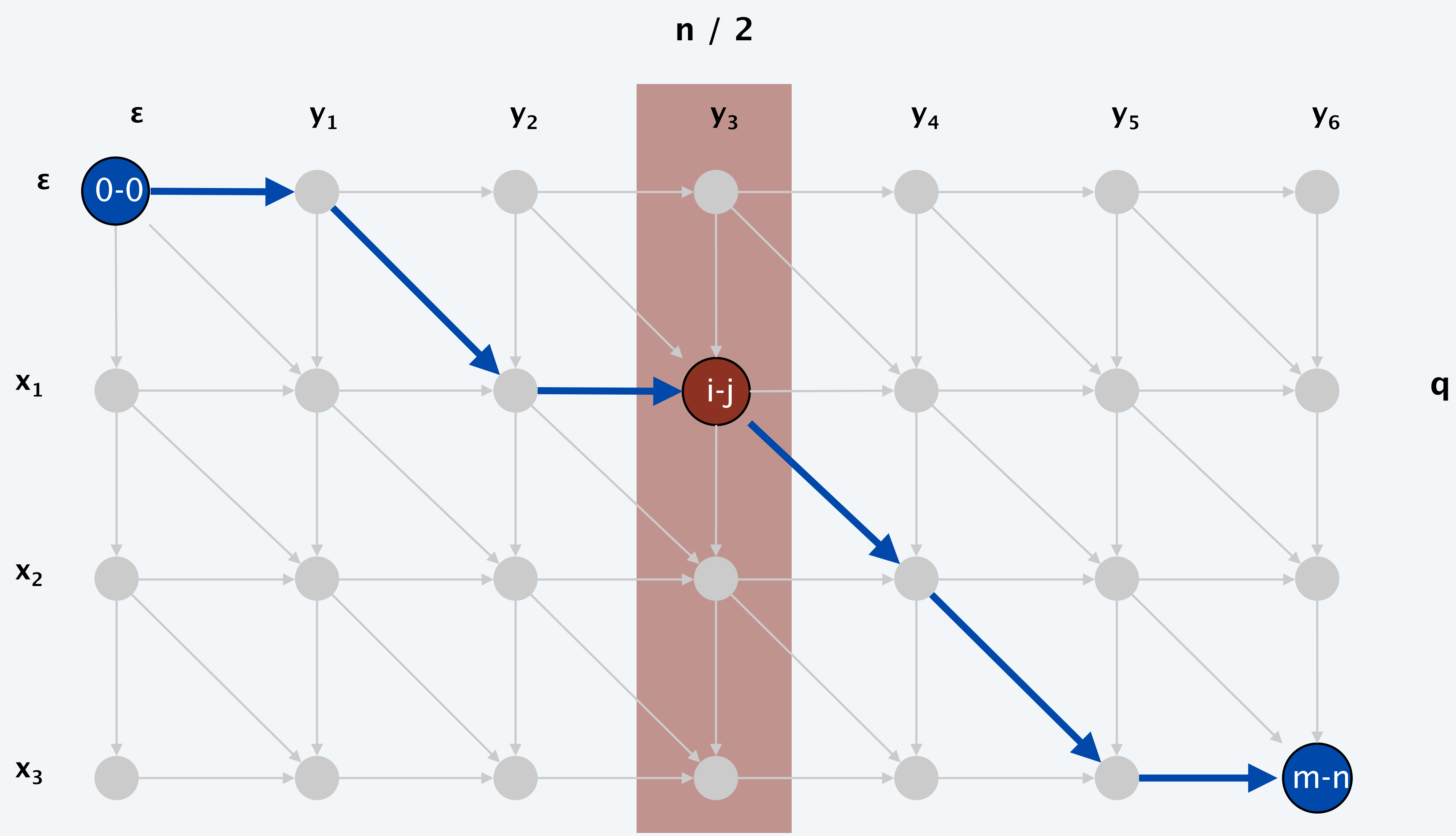
Hirschberg's algorithm

Observation 1. The cost of the shortest path that uses (i, j) is $f(i, j) + g(i, j)$.



Hirschberg's algorithm

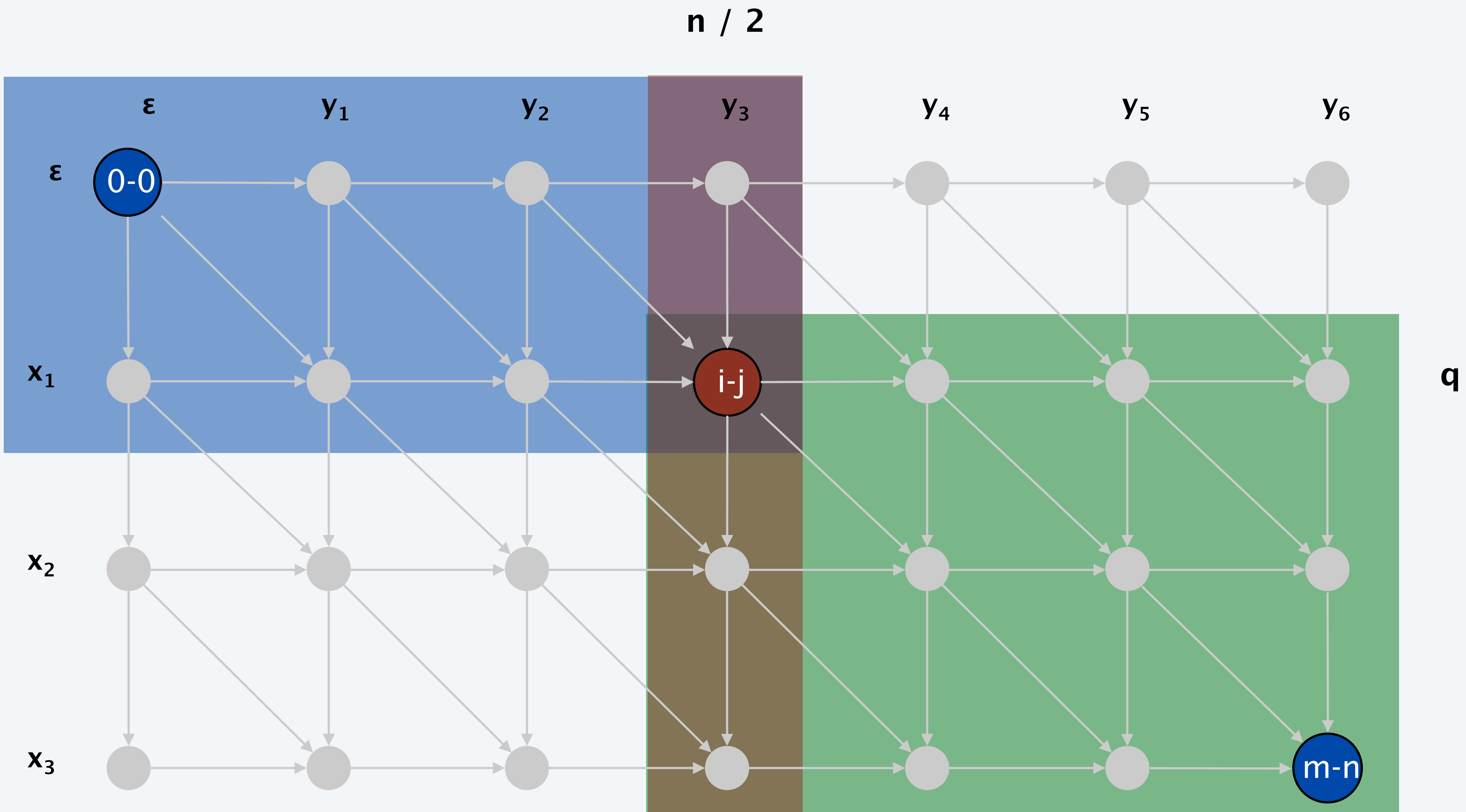
Observation 2. let q be an index that minimizes $f(q, n/2) + g(q, n/2)$.
Then, there exists a shortest path from $(0, 0)$ to (m, n) uses $(q, n/2)$.



Hirschberg's algorithm

Divide. Find index q that minimizes $f(q, n/2) + g(q, n/2)$; align x_q and $y_{n/2}$.

Conquer. Recursively compute optimal alignment in each piece.



Hirschberg's algorithm: running time analysis warmup

Theorem. Let $T(m, n)$ = max running time of Hirschberg's algorithm on strings of length at most m and n. Then, $T(m, n) = O(m\, n \log n)$.

Pf. $T(m, n) \leq 2\, T(m, n/2) + O(m\, n)$

$$\Rightarrow T(m, n) = O(m \log n).$$

Remark. Analysis is not tight because two subproblems are of size $(q, n/2)$ and $(m - q, n/2)$. In next slide, we save $\log n$ factor.

Hirschberg's algorithm: running time analysis

Theorem. Let $T(m, n)$ = max running time of Hirschberg's algorithm on strings of length at most m and n. Then, $T(m, n) = O(m n)$.

Pf. [by induction on n]

- $O(m n)$ time to compute $f(\bullet, n/2)$ and $g(\bullet, n/2)$ and find index q.
- $T(q, n/2) + T(m - q, n/2)$ time for two recursive calls.
- Choose constant c so that:

$$\begin{aligned} T(m, 2) &\leq c m \\ T(2, n) &\leq c n \\ T(m, n) &\leq c m n + T(q, n/2) + T(m - q, n/2) \end{aligned}$$

- Claim. $T(m, n) \leq 2 c m n$.
- Base cases: $m = 2$ or $n = 2$.
- Inductive hypothesis: $T(m, n) \leq 2 c m n$ for all (m', n') with $m' + n' < m + n$.

$$\begin{aligned} T(m, n) &\leq T(q, n/2) + T(m - q, n/2) + c m n \\ &\leq 2 c q n/2 + 2 c (m - q) n/2 + c m n \\ &= c q n + c m n - c q n + c m n \\ &= 2 c m n \quad \blacksquare \end{aligned}$$