Excercises from Class, part 2

April 17, 2008

In the following questions, A denotes the adjacency matrix, T denotes the diagonal degree matrix, L denotes the combinatorial Laplacian ($L = T - A$) and $\mathcal{L}$ denotes the normalized Laplacian ($\mathcal{L} = T^{-1/2} L T^{-1/2}$). The volume of a set $S \subseteq V(G)$ is defined as $\text{vol}(S) = \sum_{v \in S} d_v$.

1.) (4/9/08) Let $\sum_{i=1}^{N} a_i = N\bar{a}$. Show that

$$N \sum_{i=1}^{N} (a_i - \bar{a})^2 = \sum_{i<j} (a_i - a_j)^2.$$

2.) (4/9/08) Prove that the following ways of defining eigenvalues for the Laplacian (or, in some cases, λ_1) are equivalent.

1. For ϕ an eigenvector corresponding to λ

$$\lambda \phi = \mathcal{L} \phi$$

2. For f a harmonic eigenvector corresponding to λ:

$$\lambda f(v) = \frac{1}{d_v} \sum_{u \sim v} (f(v) - f(u)).$$

3. For ϕ an eigenvector corresponding to λ:

$$\lambda \phi(v) = \frac{1}{\sqrt{d_v}} \sum_{u \sim v} \left(\frac{\phi(v)}{\sqrt{d_v}} - \frac{\phi(u)}{\sqrt{d_v}} \right)$$

4.
$$\lambda_i = \inf_{g \perp \phi_j, j<i} \frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle}$$

5.
$$\lambda_i = \inf_{f \perp T f_j, j<i} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u f^2(u) d_u}$$

6.
$$\lambda_1 = \inf_{\substack{f \\ \sum f(v) d_v = 0}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u f^2(u) d_u}$$

7.
$$\lambda_1 = \inf_f \sup_{c \in \mathbb{R}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u (f(u) - c)^2 d_u}$$

8.
$$\lambda_1 = \inf_f \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_u (f(u) - \bar{f})^2 d_u}$$

9.
$$\lambda_1 = \text{vol}(G) \inf_f \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_{u,v} (f(u) - f(v))^2}$$

10.
$$\lambda_i = \sup_{g_0, \ldots, g_{i-1}} \inf_{g \perp g_j, j<i} \frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle}.$$

Some hints: Use problem (1) to see $(7) \Rightarrow (8)$. To show, e.g. $(4) \Rightarrow (2)$ try picking an f achieving 4, and witness what happens when one takes a new function f' with $f'(v) = f(v) + \epsilon$ (with the other entries perturbed suitably).

3.) (4/9/08) Recall that the vertices of the cube Q_n can be indexed by subsets $S \subseteq [k]$. If $S, T \subseteq [k]$, show a set of orthogonal eigenvectors of Q_n are given by
$$\phi_S(T) = (-1)^{|S \cap T|}$$

4.) (4/16/08) Write:

$$||fP^s - \pi|| = ||fT^{-1/2}((I - \mathcal{L})^s - I_0)T^{1/2}||$$

Show that $I_0 = \phi_0^* \phi_0$.

5.) (4/16/08) Recall the following definitions for the relative pointwise distance, χ-squared distance and total variation distance respectively:

$$\begin{aligned} \Delta(s) &= \max_{x,y} \frac{|P^s(y,x) - \pi(x)|}{\pi(x)} \\ \Delta'(s) &= \max_{y \in V(G)} \sqrt{\sum_{x \in V(G)} \frac{(P^s(y,x) - \pi(x))^2}{\pi(x)}} \\ \Delta_{TV}(s) &= \frac{1}{2} \max_{y \in V(G)} \sum_{x \in V(g)} |P^s(y,x) - \pi(x)| \end{aligned}$$

Prove that:

$$2\Delta_{TV}(s) \leq \Delta'(s) \leq \Delta(s)$$