Math 261
Final take home
Due: 1pm, Wednesday, June 9, 2010

You are only required to do four of the following five problems:

1. For a connected graph G on n vertices, let $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$ denote the eigenvalues of the (normalized) Laplacian. Show that for two subsets S and T, there is an edge with one endpoint in S and one endpoint in T if both S and T have volume at least $\bar{\lambda}\text{vol}(G)$ where $\bar{\lambda} = \max\{1-\lambda_1, \lambda_{n-1}-1\}$.

2. In a connected graph, let Z denote the lazy random walk $Z = (I+P)/2$ on G where P is the transition probability matrix. For a seed s (as a probability distribution) and a jumping constant α, $0 < \alpha < 1$, the PageRank $p = \text{pr}_{\alpha,s}$ is defined to be the unique vector satisfies the following recurrence:

$$p = \alpha s + (1-\alpha) p Z.$$

   For a subset S, the function f_S is defined by

$$f_S(v) = \begin{cases} \frac{d_v}{\text{vol}(S)} & \text{if } v \in S, \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that

$$\text{pr}_{\alpha, f_S}(S) \geq 1 - \frac{h(S)}{\alpha}$$

   where $h(S)$ denotes the Cheeger ratio of S.

3. Let h denote the Cheeger constant of a connected graph G on n vertices. Prove that if h is bounded below by an absolute constant than G has diameter $O(\log n)$.

4. For a connected graph G and for a real $t \geq 0$, the heat kernel of G is defined by

$$\begin{aligned} \mathcal{H}_t &= e^{-t\mathcal{L}} \\ &= I - t\mathcal{L} + \frac{t^2}{2!}\mathcal{L}^2 + \ldots \end{aligned}$$

   (a) Show that for any vertex x, we have

$$\mathcal{H}_t(x,x) \geq \frac{d_x}{\text{vol}(G)}.$$

   (b) For a subset S with $\text{vol}(S) \leq \text{vol}(G)/2$, define a function g_S by

$$g_S(v) = \begin{cases} \sqrt{\frac{d_v}{\text{vol}(S)}} & \text{if } v \in S, \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that

$$e^{-t\lambda} \geq g_S \mathcal{H}_t g_S - \frac{\text{vol}(S)}{\text{vol}(G)} \geq \left(1 - \frac{\text{vol}(S)}{\text{vol}(G)}\right) e^{-t\ h(S)}$$

   where λ denotes the first nontrivial eigenvalue of the (normalized) Laplacian and $h(S)$ is the Cheeger ratio of S.