Math 261 Midterm Take Home

Due: 12:30pm, Wednesday, May 14

1. (a) Compute the eigenvalues of a cycle C_n on n vertices.
   (b) In the graph $C_n \square C_n$, the Cartesian product of two copies of C_n, how many spanning trees are there?

2. (a) Compute the Dirichlet eigenvalues of a path P_n on n vertices (as a subgraph of C_{n+1} with Dirichlet boundary condition).
   (b) Compute the Dirichlet eigenvalues of a square grid (which is the Cartesian product $P_n \square P_n$ as a subgraph in $C_{n+1} \square C_{n+1}$ with Dirichlet boundary condition).
   (c) In the square grid, we first color a vertex blue and color its antipodal vertex red. For any vertex x, what is the probability $p(x)$ that a random walk starting at x hits the blue vertex before hitting the red vertex?
   (Hint: Use Dirichlet eigenvalues.)

3. (a) What are the eigenvalues and eigenvectors of the hypercube Q_n?
   (b) Let Q'_n denote the graph (as a folded cube) with 2^{n-1} vertices. Each vertex can be represented by a binary string of length n or its complement. Namely, each vertex is labelled by a pair $\{a_1 a_2 \ldots a_n, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n\}$. (Here $\bar{a}_i = 1 - a_i$.) Two vertices in Q'_n are adjacent if there are some representatives of the two vertices of Hamming distance 1. What are the eigenvalues and eigenvectors of Q'_n?

4. Suppose we have two simple graphs G and $\tilde{G}$. We say $\tilde{G}$ is a covering of G if there is an onto edge-preserving mapping $\pi : V(\tilde{G}) \to V(G)$ satisfying the following:
   (1) For every $\{u, v\} \in E(G)$, we have
   $$|\{\{x, y\} \in E(\tilde{G}) : \pi(x) = u, \pi(y) = v\}| = m.$$
   (2) For $x, y \in V(\tilde{G})$ with $\pi(x) = \pi(y) = u$, and v adjacent to u in G, we have
   $$|N(x) \cap \pi^{-1}(v)| = |N(y) \cap \pi^{-1}(v)|.$$
   (a) Prove that an eigenvalue of G is an eigenvalue of $\tilde{G}$.
   (b) Give an example of an eigenvalue of $\tilde{G}$ which is not an eigenvalue of G.
   (c) Under what condition an eigenvalue of $\tilde{G}$ is an eigenvalue of G?