Math 261
Midterm take home
Due: 1pm, Wednesday, May 12, 2010

You are only required to do three of the following four problems:

1. For two graphs $G = (V, E)$ and $H = (V', E')$, the box-product graph $G \Box H$ denote the graph with vertex set $\{(u, v) \,:\, u \in V, v \in V'\}$ and edge set $\{(u, u'), (v, v')\} \,:\, (u = v$ and $\{u', v'\} \in E')$ *or* $(u' = v'$ and $\{u, v\} \in E)\}$. For the cycle C_n on n vertices, compute the number of spanning trees in $C_n \Box C_n$. (Find a simple expression for your answer, if possible.)

2. In a connected graph G, we consider a subset of vertices, denoted by S. Suppose the induced subgraph on S is connected. The Green function $\mathcal{G}_S$ is defined to be the inverse of the Laplacian $\mathcal{L}_S$. Recall that $\mathcal{L} = I - D^{-1/2} A D^{-1/2}$. Now, we consider a cycle C_{n+1} with vertices $0, 1, \ldots, n$ and vertex j is adjacent to vertex $j+1$ for all j modulo $n+1$. Choose $S = \{1, 2, \ldots, n\}$. Show that for $x \leq y$,
$$\mathcal{G}_S(x, y) = \frac{2}{n+1} x(n + 1 - y).$$

3. For a connected induced subgraph S of a graph G and for a real $t \geq 0$, the Dirichlet heat kernel of S is defined by
$$\begin{aligned} \mathcal{H}_t &= e^{-t\mathcal{L}_S} \\ &= I - t\mathcal{L}_S + \frac{t^2}{2!}\mathcal{L}_S^2 + \ldots \end{aligned}$$
Show that
$$\mathcal{G}_S(x, y) = \int_0^\infty \mathcal{H}_t(x, y) dt.$$

4. In a connected graph G, let Z denote the lazy random walk $Z = (I + P)/2$ where P is the transition probability matrix. For a seed s (as a probability distribution) and a jumping constant α, $0 < \alpha < 1$, the PageRank $p = \mathrm{pr}_{\alpha,s}$ is defined to be the unique vector satisfies the following recurrence:
$$p = \alpha s + (1 - \alpha) p Z.$$
For $\beta > 0$, we consider the α-Green function to be the inverse of $\mathcal{L}_\beta = \beta I + \mathcal{L}$. Namely,
$$\mathcal{L}_\beta \mathcal{G}_\beta = \mathcal{G}_\beta \mathcal{L}_\beta = I.$$
Show that
$$\mathrm{pr}_{\alpha,s} = \beta s D^{-1/2} \mathcal{G}_\beta D^{1/2}$$
where $\beta = 2\alpha/(1 - \alpha)$.