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Electrical Networks - A Graph Theoretical Approach

1 Introduction

In this section, we will use the graph theory terminology and techniques that we have developed to examine electrical networks. In particular, we will use graph theoretical interpretations of *resistance*, *conductance*, *current*, *voltage* and view Kirchhoff's laws in light of these interpretations.

2 Kirchhoff's Laws

We can view an electrical network as a directed, weighted graph $G = (V, W)$ where $W : V \times V \to \mathbb{R}$ such that $W(u, v) \geq 0$. We can recover the underlying unweighted graph $G' = (V, E)$ by letting $E = \{\{uv\} \in V \times V | W(u, v) > 0\}$. Less formally, we take the unlabeled graph and assign a positive number to each edge, that we denote the weight of that edge, and pick a direction for each edge.

In this section, we view the weight of an edge as the *conductance* of that edge, which we will denote c_e for a given edge e. The *resistance* of an edge, r_e is defined by $r_e = \frac{1}{c_e}$. Both the resistance and conductance are independent of edge orientation, that is $r_{\{uv\}} = r_{\{vu\}}$ and $c_{\{uv\}} = c_{\{vu\}}$.

We now define *voltage* and *current*. If we have a function on vertices $f : V \to \mathbb{R}$, we can define the *voltage* or potential p_e for an edge to be the difference between values of f at its end vertices, that is $p_{u,v} := f(u) - f(v)$. We see that this immediately gives us that $p_{\{uv\}} = -p_{\{vu\}}$. We then can define current by using Ohm's Law, which states that for an edge e, the current flowing across that edge i_e is given by $i_e = \frac{p_e}{r_e} = p_e c_e$. We see that this means that $i_{\{uv\}} = -i_{\{vu\}}$, and we view negative current as positive current flowing the other way.

We can also view current as function on vertices, where

$$i_v = \begin{cases} 1 & \text{v is a } \textit{sink} \\ \text{-1} & \text{v is a } \textit{source} \\ 0 & \text{otherwise} \end{cases}$$

When we talk of current on vertices, we think of this being the current coming from an outside source, entering or leaving our network at that vertex.

In general, the values for the source and the sink need not be 1, but we will restrict ourselves to that case. We think of this as 1 unit of current entering the network at the source, and 1 unit leaving at the sink. So far, all of our definitions and rules have been for a single edge, or pair of vertices connected by an edge. A natural question then arises, how does the structure of the network determine current and voltage? This is partially answered by Kirchhoff's Laws.

Kirchhoff's Current Law

$$i_V = i_E B^*.$$

Here, i_V is the vector giving values i_v for each $v \in V$, i_E is the vector giving values i_e for each $e \in E$, and B is the directed edge adjacency matrix, the $|V| \times |E|$ matrix where

$$B_{v,e} = \begin{cases} 1 & v \text{ is the head of } e \\ \text{-1} & v \text{ is the tail of } e \\ 0 & \text{otherwise} \end{cases}$$

Kirchhoff's Current Law can also be stated as

$$i_v = \sum_{u \sim v} i_{\{uv\}},$$

and can be thought of as "what goes in must come out."

Kirchoff's Voltage Law

For any cycle $v_1, v_2, \cdots, v_k$,

$$\sum_{j=1}^{k+1} p_{\{v_j v_{j+1}\}} = 0,$$

where $v_{k+1} = v_1$.

A special case of Kirchhoff's Voltage law occurs when all edges have the same resistance, which then tells us (by Ohm's Law) that

$$\sum_{j=1}^{k+1} i_{\{v_j v_{j+1}\}} = 0,$$

Connection with the Combinatorial Laplacian

We can now see how our combinatorial Laplacian relates to electrical networks. By Ohm's Law, for given edge e we have that $i_e = p_e c_e$, and we know that $p_{\{uv\}} := f(u) - f(v)$, so we can express the vector $i_E = f_V BC$, where C is the $|E| \times |E|$ diagonal matrix of giving the conductances on each edge. But then we can use this conjunction with Kirchhoff's Current Law to state that $i_V = i_E B^* = f_V BCB^* = f_V L$, where L is the weighted combinatorial Laplacian.

3 A Simple Case

We now restrict ourselves to a specific case, where G is a wieghted graph with no loops, no multiple edges between vertices, and each edge has weight 1.

Theorem 1. *Given such a G, and $s, tinV(G)$, the source and sink, respectively, we can claim that the following current distribution satisfies Kirchhoff's laws:*

$$I_{\{uv\}} = \frac{\tau(s,a,b,t) - \tau(s,b,a,t)}{\tau(G)},$$

where $\tau(G)$ is the number of spanning trees of G, and $\tau(s,a,b,t)$ is the number of spanning trees of G such that the path running from s to t goes through a and then b.

Proof. For a given spanning tree T, we define

$$i^{(T)}_{\{ab\}} = \begin{cases} 1 & T \text{ contains a path from s to t that runs through } a \text{ and then } b \\ \text{-1} & T \text{ contains a path from s to t that runs through } b \text{ and then } a \\ 0 & \text{otherwise} \end{cases}$$

We notice that for any T, the current distribution given by $i^{(T)}_{\{ab\}}$ satisfies the Kirchhoff Current Law. Also, we notice that if any two distributions satisfy the Current Law, then their sum does as well. Therefore the sum $\sum_T i^{(T)}_{\{ab\}}$ also satisfies the Current Law, and we see it has total charge $\tau(G)$. Let $i_{\{ab\}} = \sum_T i^{(T)}_{\{ab\}} = \tau(s,a,b,t) - \tau(s,b,a,t)$. We know that the distribution $i_{\{ab\}}$ satisfies the current law, so it remains to show that is satisfies the voltage law as well. Since we assumed the resistance was 1 for each edge, we can deal with the special case.

In other words, we must show that for a cycle $v_1, v_2, \cdots, v_k$,

$$\sum_{j=1}^{k+1} i_{\{v_j v_{j+1}\}} = 0,$$

which is to show that

$$\sum_{j=1}^{k+1} \sum_T i^{(T)}_{\{v_j v_{j+1}\}} = 0.$$

To do this, we first define a *thicket*. A *thicket* is a spanning forest with exactly two components, one containing s and one containing t. For a thicket F, we define

$$i^{(F)}_{\{ab\}} = \begin{cases} i^{(F\cup\{ab\})}_{\{ab\}} & \text{if } F \cup \{ab\} \text{ is a spanning tree} \\ 0 & \text{otherwise} \end{cases}$$

Then we notice that $\sum_T i^{(T)}_{\{ab\}} = \sum_F i^{(F)}_{\{ab\}}$, and that for a fixed thicket F we have $\sum_{j=1}^{k+1} i^{(F)}_{\{v_j v_{j+1}\}} = 0$. They sum to zero since any cycle will in G will have an even number of "crossings" between the two components of F, with half of which will get weighted with a 1, and the other half will be weighted with a -1.

Then we put all these together and conclude that

$$\sum_{j=1}^{k+1}\sum_{T} i^{(T)}_{\{v_j v_{j+1}\}} = \sum_{j=1}^{k+1}\sum_{F} i^{(F)}_{\{v_j v_{j+1}\}} = \sum_{F}\sum_{j=1}^{k+1} i^{(F)}_{\{v_j v_{j+1}\}} = \sum_{F} 0 = 0.$$

Therefore the current distribution $i_{\{uv\}} = \tau(s,a,b,t) - \tau(s,b,a,t)$ satisfies both the current laws, and has current $\tau(G)$ flowing from s to t. Letting $I_{\{uv\}} = \frac{i_{\{uv\}}}{\tau(G)}$ gives our result.

$\square$

References

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