Math 262A Spring 2009 Exercises

Notice: I'm a week late at posting the exercise from May 11. Don't miss it!

May 20, 2009

1. Show that a non-negative matrix is irreducible if and only if the associated directed graph is strongly connected.

2. Let G_1, G_2 be two graphs on the same vertex set. Define $G = G_1 \cup G_2$ by taking the union of the edge sets. Then $\|G\| \leq \|G_1\| + \|G_2\|$.

3. Let A be a symmetric, non-negative matrix, and let $c_1, \ldots, c_n > 0$. We have a lemma which says that
$$\|A\| \leq \max_{i=1\ldots n} \frac{1}{c_i} \sum_{j=1}^{n} a_{ij} c_j$$
Find examples which show that each of these conditions is necessary.

4. Show that $\|K_{s,t}\| = st$, where $K_{s,t}$ is the complete bipartite graph whose parts have s and t vertices.

May 18, 2009

1. Show that a graph with the discrepancy property is also almost regular — that is: all but $o(n)$ vertices have degree $(1 + o(1))\frac{n}{2}$.

2. Show that $t_3(n, K_4^{(3)}) \geq \frac{5}{9}\binom{n}{3}$. That is: show that any 3-uniform hypergraph on n vertices with at least $\frac{5}{9}\binom{n}{3}$ edges must contain four vertices with every 3-subset of them as an edge.

May 13, 2009

1. Show that $r(k, k)$ is (asymptotically) greater than $(\sqrt{2})^k$, where $r(k, k)$ is the smallest number so that every 2-edge-coloring of the complete graph on $r(k, k)$ vertices must have a monochromatic K_k.

May 11, 2009

1. Show that, for each $\varepsilon > 0$, a random graph in $G(n, p)$ has $(p + \alpha)\binom{n}{2}$ edges, for some $|\alpha| < \varepsilon$.

April 29, 2009

1. Show that $\Delta_{TV}(s) = \frac{1}{2} \max_{y \in V} \sum_{x \in V} |P^s(y, x) - \pi(x)|$

2. Show that $\Delta(s) \leq (1 - \lambda')^s \cdot \frac{\mathrm{vol}(G)}{\min d_x}$, where $\Delta(s)$ is the relative pointwise distance.

April 22, 2009

1. If G is edge-transitive and k-regular, does it have to be vertex-transitive?

April 20, 2009

1. Find graphs where $\chi(G)$ is closely approximated by $1 + \max \frac{1}{\lambda_w - 1}$.
2. Compute h_G for $G = Q_n, P_n$.
3. Verify the Cheeger inequality for Q_n, P_n.

April 13, 2009

1. Find i_{ab} for a simple graph.
2. Suppose $f : V \to \mathbb{Z}$ satisfies $R(f) = \lambda = \lambda_1$. Construct $f' : V\ \mathbb{R}$ by $f'(x_0) = f(x_0) + \varepsilon/d_{x_0}$, and $f'(x) = f(x) - \varepsilon/(\mathrm{vol}(G) - d_{x_0})$ for all other x.
   Verify that $R(f') - R(f) = \varepsilon\left(\sum_{y\sim x_0}[f(x_0) - f(y)] - \lambda f(x_0)d_x\right) + O(\varepsilon^2)$.
3. Verify that there is some graph G and some constant c such that $\lambda_1 < \frac{c}{D\cdot\mathrm{vol}(G)}$, where D is the diameter of G.

April 8, 2009

1. (Solved in class) Explain why current flow in a graph is uniquely determined by resistances and the position of the sink and source.
2. Let G be a 5-cycle with an additonal edge, with all edges weighted 1. Pick a source and a sink for current flow. Determine the current at each edge, and draw the horizontal line graph associated with this flow.

April 6, 2009

1. Find the spectrum of P_n, the path on n vertices.

April 1, 2009

1. Find the spectrum of K_n, the complete graph on n vertices.
2. Find the eigenvalues of A, where A is the symmetric $n \times n$ matrix given by

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & \cdots & \iddots & a_0 \\ a_2 & a_3 & a_4 & \iddots & \iddots & a_1 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \iddots & \vdots \\ a_{n-1} & a_0 & a_1 & \cdots & \cdots & a_{n-2} \end{pmatrix}$$

3. Let $G = (V, E)$ be a graph, with normalized Laplacian $\mathcal{L}$. Let $\mathcal{L}_v$ be the minor of $\mathcal{L}$ formed by removing the row and column associated with the vertex $v \in V$ (ie $\mathcal{L}_v$ is the determinant of that submatrix). Let d_v be the degree of the vertex v, $0 = \lambda_0 \le \lambda_1 \le \ldots \le \lambda_{n-1} \le 2$ be the eigenvalues of $\mathcal{L}$, and $\tau(G)$ be the number of labelled spanning trees of G. Show that

$$\sum_{v\in V} \mathcal{L}_v = \frac{\sum_{v\in V} d_v}{\prod_{v\in V} d_v}\tau(G) = \prod_{i=1}^{n-1} \lambda_i$$

March 30, 2009

1. Let $G = C_5$ be the five-cycle, with vertices $0, 1, 2, 3, 4$, where vertex j has neighbors $j - 1$ and $j + 1 \pmod 5$. Show that the eigenvectors of the adjacency matrix of G are given by the vector $c_\theta(j) = \theta^j$, with eigenvalue $\theta + \theta^{-1}$. Here θ goes over each 5th root of unity.

2. Let Q_n be the n-cube, with vertex set $2^{[n]}$ — each vertex is a subset of $\{1, 2, \ldots, n\}$ — and two vertices are adjacent if they differ by one element. Show that the eigenvectors of the adjacency matrix of Q_n are given by the vectors

$$\Phi_S(X) = \frac{(-1)^{|S \cap X|}}{2^{n/2}}$$

Note this is the X coordinate of the eigenvector associated with $S \subseteq [n]$.