Necklace Lemma. *Let C be a cycle whose vertices are colored with k colors. Then one can remove k edges so that the remaining connected components can be partitioned into two sets, each of which contains equinumerous vertices (to within one) in each color.*

We omit the proof of the above lemma, first proven by Goldberg and West. This nontrivial result relies on the Borsuk-Ulam theorem, which states that any continuous map $f : S^n \to \mathbb{R}^n$ identifies a pair of antipodal points.

Recall that any forest *F* having *n* vertices can be separated into two parts, F_{v0} and F_{v1}, each of which has fewer than $\frac{2}{3}n$ vertices, by removing a single vertex *v*. We refer to such a vertex as a *splitting vertex*, and we say that F_{v0} and F_{v1} are *split by* v.

Lemma 1. *Let T be a tree on n vertices, and let C be a complete binary tree having* $\left\lceil \frac{\log n}{\log 3/2} \right\rceil$ *levels. Then there exists a map* $f : V(T) \to V(C)$ *satisfying the following three properties.*

- **(i)** *f is injective.*
- **(ii)** *For each vertex w in C, let* S_w *denote the set of vertices in T that are mapped by f to descendents of w. Let* w_0 *and* w_1 *be the descendents of w satisfying* $|S_{w0}| \le |S_{w1}|$, $|S_{w1}| > 0$. *Then* $|S_{w1}| < 2|S_{w0}|$.
- **(iii)** *For each w in C, let* A_w *denote the set of vertices in T that are mapped by f to ancestors of w or to w itself. Then* S_w *is separated from the rest of the graph in* $T \setminus A_w$.

Proof. Let *v* be a splitting vertex in *T* (chosen arbitrarily among all splitting vertices). Define $f(v)$ to be the root of *C*. As above, let F_{v0} and F_{v1} be the forests (having fewer than $\frac{2}{3}n$ vertices) that are split by *v*, with $|V(F_{v0})| \le |V(F_{v1})|$. We let the left descendents of $f(v)$ contain $f(F_{v0})$ and the right descendents of $f(v)$ contain $f(F_{v1})$. The restrictions $f|_{V(F_{v0})}$ and $f|_{V(F_{v1})}$ are then defined inductively.

Note that two adjacent vertices in *T* might not be mapped to adjacent vertices in *C*. In fact, since the maximum degree of any complete binary tree is at most 3, if any vertex *v* of *T* has degree at least 4 then one of its neighbors will not be mapped to a vertex adjacent to $f(v)$.

The upper bound on the number of levels of *C* follows immediately from the inductive definition of *f* together with the fact that *v* is a splitting vertex. Also, f is injective by its inductive definition while **(ii)** holds because *w* is a splitting vertex.

To prove **(iii)**, consider two vertices u_1 and u_2 which are adjacent in *T*. Suppose neither $f(u_1)$ nor $f(u_2)$ is a descendent of the other. Let $w \in V(C)$ be the common ancestor of $f(u_1)$ and $f(u_2)$ for which $d_C(w, f(u_1))$ is minimized. Then one of $f(u_1)$ or $f(u_2)$ is a left descendent of *w* while the other is a right descendent. But by definition