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Math 262 Lectures Notes

Large Deviation Inequalities

1 An Alternate Model

Before we go into large deviation inequalities, we first introduce an alternate random graph model, the weighted graph model $G(w)$. Here we consider a weight vector $w = (w_1, w_2, \ldots, w_n)$ where each $w_i \geq 0$. In particular, we will let w_k be the expected degree at vertex k. Then $Pr\big(\{i,j\} \text{ is an edge}\big) = \frac{w_i w_j}{\sum_{k=1}^n w_k}$. Thus we need to place some sort of restriction so that each $\frac{w_i w_j}{\sum_{k=1}^n w_k} < 1$. For instance, we can restrict each $w_i < \sqrt{\sum_k w_k}$. So now if we let $X =$ the degree of some fixed v_i, then $X = \sum_{j=1}^n X_{i,j}$, where $X_{i,j} = 1$ if $\{i,j\}$ is an edge and $X_{i,j} = 0$ otherwise. Thus:

$$E(X) = \sum_j E(X_{i,j}) = \sum_j \frac{w_i w_j}{\sum_k w_k} = w_i$$

If we look at the special case that each $w_i = np$ for some $0 < p < 1$, then this model reduces to the familiar $G(n,p)$ model. Also we note that here (and throughout the rest of this lecture) we allow loops.

2 Two Large Deviation Inequalities

2.1 First Large Deviation Inequality

Let $X = X_1 + X_2 + \ldots + X_m$, where the X_i's are mutually independent indicator random variables. We will assume that $Pr(X_i = 1) = p_i$ and $Pr(X_i = 0) = 1 - p_i$, where each p_i is between 0 and 1. Thus, $E(X) = \sum i = 1^m p_i$. Our first large deviation inequality is:

$$Pr\big(|X - E(X)| > t\big) < 2\exp\left(-\frac{t^2}{2E(X) + \frac{2}{3}t}\right)$$

2.2 Second Large Deviation Inequality

This is the generalized martingale inequality with which we are already familiar (see notes from the last two weeks). Here is a brief recap: Here we have a sequence $X_0, X_1, \dots, X_m = X$ and a nonnegative vector $c = (c_1, c_2, \dots, c_m)$. The vector c gives us a c-Lipschitz condition: $|X_i - X_{i-1}| \leq c_i$. So we have the following inequality:

$$Pr\big(|X - E(X)| > t\big) < 2\exp\left(-\frac{t^2}{2\sum c_i^2}\right) + Pr(B)$$

where $Pr(B)$ represents the probability of following a “bad path” (i.e., the c-Lipschitz condition is violated).

3 Applications of the Large Deviation Inequalities

3.1 Maximum Degree in G(n,p)

We will look at $G(n, p)$ and choose $p = c/n$ for some constant c. This is definitely a sparse graph model! We claim that the maximum degree is $\leq (\sqrt{2/3}+\epsilon)\log n$ with probability 1 as n approaches infinity, where $\epsilon > 0$ is as small as we wish. We pick $t = (\sqrt{2/3} + \epsilon)\log n$. Let v be any vertex and d be its degree. Then since $np = c$ becomes negligible compared with d and with $\log n$ as n approaches infinity, we have from the first large deviation inequality:

$$Pr\left(d > \left(\sqrt{2/3}+\epsilon\right)\log n\right) \leq Pr\left(|d-np| > \left(\sqrt{2/3}+\epsilon\right)\log n\right) < 2\exp\Big(-(1+\delta)\log n\Big) = 2n^{-1-\delta}$$

where $\delta = \sqrt{6}\epsilon + \epsilon^2$. Thus:

$$Pr\left(\text{some } v \in V(G) \text{ has } d > (\sqrt{2/3}+\epsilon)\log n\right) \leq \sum_{v\in V(G)} Pr\left(d(v) > (\sqrt{2/3}+\epsilon)\log n\right) \leq (2n)(n^{-1-\delta}) = 2n^{-\delta}$$

The right hand term approaches 0 as n approaches infinity, and our desired result follows.

3.2 Maximum Codegree

In $G(n,p)$, fix two vertices u and v. Let $X_{u,v} =$ the number of common neighbors of u and v. Then for any given vertex w, $Pr(w$ is a common neighbor$) = p^2$. So $E(X_{u,v}) = np^2$. We claim now that if we now let $p = 1/\sqrt{n}$ (we are looking now at a denser graph than in the previous example) then any pair of vertices will have less than $(4/3+\epsilon)\log n$ common neighbors as n increases without bound, where $\epsilon > 0$ is as small as we wish. We once again use the first large deviation inequality. We choose $t = (4/3+\epsilon)\log n$. Then for any given u and v, $Pr\big(|X_{u,v} - E(X_{u,v})| > t\big) < 2\exp\big((-2-(3/2)\epsilon)\log n\big) = 2n^{-2-\delta}$, where $\delta > 0$. So the probability that any u and v have more than t common neighbors is less than or equal to:

$$\sum_{u,v} Pr\left(|X_{u,v} - E(X_{u,v})| > t\right) < (2n^2)(n^{-2-\delta}) = 2n^{-\delta}$$

The right hand side approaches zero as n approaches infinity, and the result follows.

3.3 Counting Triangles

Here we use the results from the previous section, along with the generalized martingale inequality. In $G(n,p)$ let X be the number of triangles. As the triangles are not independent, our first inequality does not apply. Now $E(X) = n^3p^3$. So we will try letting $p = 1/\sqrt{n}$. And let $t = n^{3/2}$. From our previous result, we will here let our Lipschitz constants $c_i = 2\log n$, where we let $\epsilon = 1/3$ to avoid messiness. We also see from previous result that $Pr(B) << n^2 e - t$. So $Pr(B)$ approaches 0 as n approaches infinity. So:

$$Pr\left(|X - n^3p^3| > t\right) \leq 2\exp\left(\frac{-t^2}{8\binom{n}{2}\log^2 n}\right) + Pr(B)$$

But since $t = n^{3/2}$, the right hand side approaches 0 as n approaches infinity. So we see that we almost certainly have less than $n^{3/2}$ triangles.