Notes on constructing Folkman graphs with spectral techniques

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May 13, 2011

All material below comes from Lincoln Lu’s paper “Explicit Construction of Small Folkman Graphs”

1 Introduction

For a graph \( G = (E, V) \), a \( c \)-edge-coloring of a graph is a function \( \chi : E \to \{1, \ldots, c\} \). Here the “colors” are the numbers 1, \ldots, \( c \). Ramsey’s theorem says that, for every \( c, k \in \mathbb{N} \), there is a number \( N = R(k) \) so that every \( c \)-edge-coloring of \( K_N \) contains a monochromatic \( K_k \) — a subgraph on \( k \) vertices, all of whose edges are the same color. This is concisely written as \( K_N \to (K_k)_c \). A popular special case of Ramsey’s theorem states that \( K_6 \to (K_3)_2 \).

Folkman showed that, when looking for a monochromatic triangle, a large clique is not actually needed: for every \( c \), there is a graph \( G \) which contains no \( K_4 \) so that \( G \to (K_3)_c \). (For \( c = 2 \), such a graph is called a Folkman graph). In general Nešetřil and Rödl showed that, for all \( c, k \), there is a \( K_{k+1} \)-free graph \( G \) so that \( G \to (K_k)_c \). In that spirit, define the function \( f(c, k, \ell) \) (with \( \ell > k \)) to be the least number of vertices in a \( K_{\ell} \)-free graph \( G \) with \( G \to (K_k)_c \).

It is known that \( f(2, 3, 6) = 8 \) (by removing a 5-cycle from an 8-clique), and \( f(2, 3, 5) = 27 \). \( f(2, 3, 4) \), the size of the smallest Folkman graph was originally proven to be less than 700 billion, further reduced to 3 billion. We will find a Folkman graph on 9697 vertices, answering a problem of Erdős to show \( f(2, 3, 4) \leq 1,000,000 \).

We will first give a spectral condition for a graph to have \( G \to (K_3)_2 \). We will then give a family of graphs for which this condition is particularly easy to check. From here the process of finding a small Folkman graph is easy to automate, so we simply give the results.

2 Counting edges to show \( G \to (K_3)_2 \)

Suppose \( G \to (K_3)_2 \). Then, for some particular coloring \( \chi : E \to \{\text{red, blue}\} \), every triangle contains at least one red edge, and at least one blue edge. Viewed another way, every triangle \( xyz \) has a vertex \( x \) so that \( xy \) is red, and \( xz \) is blue. In fact, every triangle has exactly two of these vertices. We may summarize this by saying

\[ |\{xyz : xy \text{ is a red edge, } xz \text{ is a blue edge, } yz \text{ is an edge}\}| = 2 \cdot (\# \text{ of triangles in } G). \]

The existance of a coloring satisfying the above is equivalent to \( G \to (K_3)_2 \).

So, if every coloring leads to a monochromatic triangle, then the left hand side would always be strictly less then twice the number of triangles. That is, \( G \to (K_3)_2 \) is equivalent to

\[ |\{xyz : xy \text{ is a red edge, } xz \text{ is a blue edge, } yz \text{ is an edge}\}| < 2 \cdot (\# \text{ of triangles in } G) \]

for every edge-coloring \( \chi \).
Writing \(e(X,Y)\) as the number of edges between two disjoint sets of vertices \(X\) and \(Y\), the left hand side is equal to \(\sum_{v \in V} e(\text{red neighbors of } v, \text{blue neighbors of } v)\), where we say a neighbor of \(v\) is red if they are connected by a red edge, and likewise for blue. Meanwhile, the number of triangles in a graph may be seen as \(\frac{1}{3} \sum_{v \in V} |E(\text{neighbors of } v)|\), since the sum counts each triangle three times (once at each of its vertices). This tells us that \(G \to (K_3)_2\) is equivalent to
\[
\sum_{v \in V} e(\text{red neighbors of } v, \text{blue neighbors of } v) < \frac{1}{3} \sum_{v \in V} |E(\text{neighbors of } v)|.
\]
for all edge-colorings. To check that \(G \to (K_3)_2\), it is therefore sufficient to compare the corresponding terms. Writing the induced graph on the neighbors of \(v\) as the local graph \(G_v\), and taking a maximum over all possible partitions of the neighbors of \(v\) into color classes, a sufficient condition is that, for all \(v \in V\),
\[
b(G_v) := \max\{e(X,Y) \mid X,Y \subset V(G_v) \text{ disjoint}\} < \frac{2}{3} |E(G_v)|
\]
Consider the function \(b(H)\) for a fixed graph \(H\). By taking a random partition of the vertices, we see that \(b(H) \geq \frac{1}{2} |E(H)|\).

**Definition:** For a graph \(H\), and \(0 < \delta < \frac{1}{2}\), call \(H\) \(\delta\)-fair if \(b(H) < (\frac{1}{2} + \delta)|E(H)|\).

What we have shown so far may be concisely written in this terminology. For a graph \(G\), if the local graph for each vertex \(v\) is \(\frac{1}{6}\)-fair, then \(G \to (K_3)_2\). If \(G\) is also \(K_4\)-free, then it is a Folkman graph. The key point here is that

### 3 Spectral conditions for \(G \to (K_3)_2\)

We now show how to bound how fair a graph is.

Fix a graph \(H\). Let \(A\) be its adjacency matrix, and \(\vec{d} = A \cdot \vec{1}\) be the vector of degrees. Let \(d = \frac{\text{vol} H}{n}\) be the average degree. Define the matrix
\[
M = A - \frac{1}{\text{vol} H} \vec{d} \cdot \vec{d}^t.
\]

**Lemma 3.1** If the smallest eigenvalue of \(M\) is greater than \(-2\delta d\), then \(H\) is \(\delta\)-fair.

**Proof:** Let \(X \subseteq V(H)\), and \(Y = \bar{X}\). Note that \(\vec{1}_X + \vec{1}_Y = \vec{1}\).

Observe that \(\vec{1}\) is an eigenvector of \(M\), with eigenvalue 0, since
\[
M \vec{1} = A\vec{1} - \frac{1}{\text{vol} H} \vec{d} \cdot \vec{d}^t \vec{1} = \vec{d} - \frac{1}{\text{vol} H} \vec{d} \cdot \text{vol} H = \vec{0}.
\]

For \(t \in (0,1)\), define \(\vec{a}(t) = (1-t)\vec{1}_X - t\vec{1}_Y\). Note that
\[
\vec{a}(t) = \vec{1}_X - t\vec{1} = \vec{1}_Y + (1-t)\vec{1}.
\]

We see that
\[
\vec{a}(t)^* M \vec{a}(t) = (\vec{1}_X - t\vec{1}) \cdot M \cdot (\vec{1}_Y + (1-t)\vec{1}) = \vec{1}_X M \vec{1}_Y
\]
\[
= -\vec{1}_X A \vec{1}_Y + \frac{1}{\text{vol} H} \vec{1}_X \cdot \vec{d} \cdot \vec{d}^t \cdot \vec{1}_Y
\]
\[
= -e(X,Y) + \frac{\text{vol} X \text{vol} Y}{\text{vol} H}.
\]
Now we consider $\rho$, the smallest eigenvalue for $M$. By assumption, $\rho < -2\delta d$. This means that, for all vectors $\vec{v}$, $\vec{v}^T M \vec{v} \geq \rho \|\vec{v}\|$. Applying this to $\vec{\alpha}(t)$, we see

$$-e(X,Y) + \frac{\text{vol } X \text{vol } Y}{\text{vol } H} \geq \rho \|\vec{\alpha}(t)\|.$$ 

Optimizing at $t = \frac{|X|}{n}$ (so that $\vec{\alpha}(t)$ is orthogonal to $\vec{1}$), and then applying Cauchy-Schwarz, this tells us

$$e(X,Y) < \frac{\text{vol } X \text{vol } Y}{\text{vol } H} + 2\delta d \frac{|X||Y|}{n^2} + \frac{4\delta d |X||Y|}{4n^2} < \frac{\text{vol } H}{4} + 2\delta d \frac{n}{4} < \frac{1}{2} + \delta \frac{\text{vol } H}{2} = \left(\frac{1}{2} + \delta\right)|E|.$$ 

Thus $H$ must be $\delta$-fair. ■

As a corollary, we see that if $H$ is $d$-regular, it suffices that the minimum eigenvalue of $A$ is larger than $2\delta d$. To see this, simply note that $M = A - \frac{d}{4} J$ (where $J$ is the all ones matrix). Where $1$ has eigenvalue $0$ for $M$, it has eigenvalue $d$ for $A$. Meanwhile, the difference between $A$ and $M$ is simply an adjustment by an orthogonal projection map onto $\vec{1}$, so all other eigenvectors $\vec{\phi}$ have $J\vec{\phi} = 0$, and are thus unaffected by this shift.

Going back to the quest for Folkman graphs, we “just” need that, for every vertex $v$, if $G_v$ is $d$-regular, its adjacency matrix should have

$$\sigma := \frac{\lambda_{\text{min}}}{\lambda_{\text{max}}} > -\frac{1}{3},$$

since $\lambda_{\text{max}} = d$.

### 4 An easy-to-check family of graphs

Consider the Cayley graph generated by $S \subset \mathbb{Z}_m$. That is, $G$ has vertex set $\mathbb{Z}_m$, and $\{x,y\}$ form an edge if and only if $x - y \in S$. In order to form an undirected, loop-less graph, we must assume that $0 \notin S$, and $S$ is closed under negatives.

It is a simple exercise to check that the eigenvalues of the adjacency matrix of such a graph are given by

$$\left\{ \sum_{s \in S} \cos \left( \frac{2\pi ks}{n} \right) \mid k = 0, \ldots, n-1 \right\}.$$

(It is helpful to view the adjacency matrix as $\sum_{s \in S} C^s$, where $C\vec{e}_i = \vec{e}_{i+1}$, taking the indices mod $m$).

We would like a graph $G$ so that each of its local graphs is such a Cayley graph. It turns out that we may take $G$ itself to have the same form! For a given $s$ relatively prime to $m$, define

$$S(s) = \{s^k \in \mathbb{Z}_m \mid k \in \mathbb{N}\},$$

and define $L(m, s)$ to be the graph on $\mathbb{Z}_m$ generated by $S(s)$, so long as $-1 \in S(s)$. Since we know $s^n \equiv 1 \mod m$ for some $n$, $S(s)$ is itself a cyclic group under multiplication mod $m$. Consider a local graph of $G = L(m, s)$. Since Cayley graphs are vertex-transitive, it suffices to consider the local graph $G_v$ when $v = 0$. Its vertices, the neighbors of $0$, are precisely $S$. If we define $f : \mathbb{Z}_n \to S$ by $f(i) = s^i$ (an isomorphism of groups), it is easy to check that the Cayley graph on $\mathbb{Z}_n$ generated
by $T = \{i \mid f(i) - 1 \in S\}$ is isomorphic to $G_v$ under the isomorphism $f$. Thus, given $m, s$, we (with the help of a computer) may easily compute $T$, and determine the eigenvalues of $G_v$.

Lincoln Lu did just that, directing Maple to compute the eigenvalues for the local graphs of all graphs $L(m, s)$ with $m$ relatively small — he gives results up to $m = 57401$. For those satisfying the eigenvalue condition, he also checked whether they contained a $K_4$. As it turns out, $L(9697, 4)$ passes both tests, and is therefore a Folkman graph on 9697 vertices. In fact, Lu states that any 2-edge-coloring of $L(9697, 4)$ contains a monochromatic $K_4$ minus an edge.

5 Further questions

Lu mentions that Exoo checked some smaller graphs, and conjectured that $L(127, 5)$ is a Folkman graph, despite failing the eigenvalue condition. Is there a stronger condition which helps to decide this? Is it possible to modify these arguments to find a small $K_4$-free graph so that every 3-edge-coloring gives a monochromatic triangle?