Matchings in Regular Graphs from Eigenvalues

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May 3, 2011
$G(V, E)$ a simple graph with $|V| = n$, $|E| = e$. $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ the eigenvalues of $A$. 

**Uses of Eigenvalues of $A$**

- $\mu_1$: Chromatic Number, Independence Number, Clique Number
- $\mu_2$: Expansion Properties
- $\mu_n$: Chromatic Number, Independence Number, Maximum Cut
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Matching Number

- $\nu(G) \coloneqq$ maximum size of a matching of $G$. 

Theorem (Perfect Matching Theorem)

Let $G$ be a connected, $k$-regular graph. If $\mu_3 < \rho(k)$, then $\nu(G) = \lfloor \frac{n}{2} \rfloor$.

Theorem (Factor Critical Theorem)

Let $G$ be a connected, $k$-regular graph, with $n$ odd. If $\mu_2 < \rho(k)$, then $G$ is factor critical (i.e., $\forall x \in V, G \setminus \{x\}$ has a perfect matching).
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- $\nu(G) :=$ maximum size of a matching of $G$.
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$\mathcal{H}(k) =$ set of graphs $G$ such that:

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**Definition**

$$\rho(k) := \min_{G \in \mathcal{H}(k)} \mu_1(G)$$
Computing $\rho$

Theorem

$$\rho(k) := \begin{cases} 
\theta & \text{if } k = 3 \\
\frac{1}{2} \left( k - 2 + \sqrt{k^2 + 12} \right) & \text{if } k = 2j, j \geq 2 \\
\frac{1}{2} \left( k - 3 + \sqrt{(k + 1)^2 + 16} \right) & \text{if } k = 2j + 1, j \geq 2 
\end{cases}$$

where $\theta$ is the maximum root of the polynomial $x^3 - x^2 - 6x + 2$. 
Lemma

Let $G$ be a connected, $k$-regular graph with $k \geq 3$. If $\nu(G) \leq \frac{n-2}{2}$, then $G$ has three vertex disjoint induced subgraphs $H_1, H_2, H_3 \in \mathcal{H}(k)$.

Proof.

We consider the case where $k$ is even. The Berge-Tutte Formula tells us that

$$\nu(G) = \frac{1}{2}(n + \min_{S \subseteq V} (|S| - odd(G \setminus S)))$$

where $odd(H)$ is the number of odd components of $H$. 
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where $odd(H)$ is the number of odd components of $H$. Suppose that $\nu(G) \leq \frac{n-2}{2}$. Then there exists $S \subset V$ such that

$$2\nu = n + s - q,$$

where $s = |S|$, $q = odd(G \setminus S)$. 
Proof of Lemma

Proof.

Note that $q \geq s + 2$, and $s > 0$.
Let $H_1, \ldots, H_q$ be the odd components of $G \setminus S$, where $H_i$ has $n_i$ vertices and $e_i$ edges. Let $t_i = |E(H_i, S)|$. $G$ is connected so $t_i \geq 1$. Since vertices in $H_i$ are adjacent to those in $H_i$ or $S$, we get that

$$2e_i = kn_i - t_i = k(n_i - 1) + k - t_i.$$
Proof.

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$$2e_i = kn_i - t_i = k(n_i - 1) + k - t_i.$$

Since $n_i$ is odd, we get that $k - t_i$ is even, so $k$ and $t_i$ have the same parity for all $i$. 

We claim there are at least 3 $i$'s such that $t_i < k$. Suppose not. Then there are $q - 2$ $i$'s such that $t_i \geq k$. This implies that

$$\text{vol}(S) = ks \geq \sum i t_i \geq k(q - 2) + 2 \geq ks + 2,$$

a contradiction. WLOG denote these $H_1, H_2, H_3$. 

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Note that \( q \geq s + 2 \), and \( s > 0 \).

Let \( H_1, \ldots, H_q \) be the odd components of \( G \setminus S \), where \( H_i \) has \( n_i \) vertices and \( e_i \) edges. Let \( t_i = |E(H_i, S)| \). \( G \) is connected so \( t_i \geq 1 \). Since vertices in \( H_i \) are adjacent to those in \( H_i \) or \( S \), we get that

\[
2e_i = kn_i - t_i = k(n_i - 1) + k - t_i.
\]

Since \( n_i \) is odd, we get that \( k - t_i \) is even, so \( k \) and \( t_i \) have the same parity for all \( i \).

We claim there are at least 3 \( i \) such that \( t_i < k \). Suppose not. Then there are \( q - 2 \) \( i' \)'s such that \( t_i \geq k \). This implies that

\[
\text{vol}(S) = ks \geq \sum_i t_i \geq k(q - 2) + 2 \geq ks + 2,
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a contradiction. WLOG denote these \( H_1, H_2, H_3 \).
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Proof.

Now since \( t_i \) and \( k \) have the same parity, those \( t_i \leq k - 2 \). Since \( t_i \leq k - 2 \) we get that

\[
2e_i = kn_i - t_i \geq kn_i + 2 - k,
\]

but since

\[
n_i(n_1 - 1) \geq 2e_i,
\]

we have that

\[
n_i \geq k + \frac{2}{n_i - 1}
\]

Thus \( n_i \geq k + 1 \geq t_i + 3 \). Thus each \( H_1, H_2, H_3 \) has at least 3 vertices of degree \( k \), so \( H_1, H_2, H_3 \in \mathcal{H}(k) \).
The Inclusion Principle

Lemma (The Inclusion Principle)

Let $A$ be a Hermitian matrix, and $1 \leq r \leq n$. Let $A_r$ denote any $r \times r$ principal submatrix of $A$. Then for any $j$ such that $1 \leq j \leq r$,

$$
\mu_k(A) \geq \mu_k(A_r) \geq \mu_{k+n-r}(A).
$$
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Let $G$ be a connected, $k$-regular graph. If

$$\mu_3 < \rho(k),$$

then

$$\nu(G) = \left\lfloor \frac{n}{2} \right\rfloor$$
Proof of Perfect Matching Theorem

Proof.
Suppose $G$ satisfies the condition of the Perfect Matching Theorem and $\mu_3 < \rho(k)$. Suppose, for contradiction that $\nu(G) \leq \frac{n-2}{2}$. Then by Lemma 1, $G$ has three vertex disjoint induced subgraphs $H_1, H_2, H_3 \in \mathcal{H}(k)$. Thus by the inclusion principle

\[ \mu_3(G) \geq \mu_3(H_1 \cup H_2 \cup H_3) \geq \min_i \mu_1(H_i) \geq \rho(k), \]

a contradiction. Thus $\nu(G) > \frac{n-2}{2}$. 

A Generalization

**Theorem**

Let $G$ be a connected, $k$-regular graph. If

$$\mu_r \leq \rho(k),$$

for $3 \leq r < n$, then

$$\nu(G) > \frac{n - r + 1}{2}$$
Exercises and Research Problems

- Prove the Inclusion Principle
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Hints: Use either the interlacing inequalities, or Courant-Fisher
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- Can this theorem be applied meaningfully? Find a non-trivial family of graphs where one has good control on $\mu_3$. 

All the proofs were for regular graphs, what can be said if the graph is "nearly regular", e.g. if $|d_v - k| \leq c$ for some small $c$. 

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