1 Spectral Techniques in Extremal Graph Theory

Two of the most fundamental results in extremal graph theory are Turán’s Theorem and the Erdős-Stone Theorem. Turán’s Theorem tells us that if an \( n \)-vertex graph \( G \) has more than \( (1 - \frac{1}{r} + o(1)) \frac{n^2}{2} \) edges, then \( K_{r+1} \subset G \). The Erdős-Stone Theorem proves that for large enough \( n \), not only do we get a \( K_{r+1} \) subgraph, but a \( K_{r+1}(t) \) subgraph (a complete \( r+1 \)-partite graph with \( t \) vertices in each part) for arbitrarily large \( t \).

**Theorem 1.1 (Erdős, Stone [5])** Let \( \epsilon > 0 \) and let \( r, t \geq 1 \) be integers. There exists an \( n_0 = n_0(\epsilon, r, t) \) such that if \( G \) is any \( n \)-vertex graph with \( n \geq n_0 \) and

\[
e(G) > (1 - \frac{1}{r} + \epsilon) \frac{n^2}{2}
\]

then \( K_{r+1}(t) \subset G \).

Bollobás and Erdős proved a strengthening of the Erdős-Stone Theorem by establishing a good lower bound on how big one can expect \( t \) to be in terms of \( r \) and \( \epsilon \).

**Theorem 1.2 (Bollobás, Erdős [1])** Let \( \epsilon > 0 \) and let \( r \geq 1 \) be an integer. There exists an \( n_0 = n_0(\epsilon, r) \) such that if \( G \) is any \( n \)-vertex graph with \( n \geq n_0 \) and

\[
e(G) > (1 - \frac{1}{r} + \epsilon) \frac{n^2}{2}
\]

then \( K_{r+1}(t) \subset G \) for some \( t \geq \epsilon \log n / (2^{r+1}(r-1)! \).

It would be nice to have spectral versions of such theorems. Fan Chung [3] proved a spectral version of Turán’s Theorem improving upon a result of Sudakov, Szabó, and Vu [12]. Recently Nikiforov [10] has obtained a spectral version of Theorem 1.2 by combining classical extremal techniques with spectral techniques. Our purpose is to describe some of the spectral techniques that played a role in Nikiforov’s proof. First some notation.

Given a graph \( G \), let \( k_r(G) = k_r \) be the number of \( r \)-cliques in \( G \) and let \( k_r(v) \) be the number of \( r \)-cliques that contain vertex \( v \). A \( k \)-walk is a sequence of vertices \( v_1, \ldots, v_k \) such that \( v_i \) is adjacent to \( v_{i+1} \) for \( 1 \leq i \leq k-1 \). Let \( w_k(G) = w_k \) be the number of \( k \)-walks in \( G \) and let \( w_k(v) \) be the number of \( k \)-walks in \( G \) that start at \( v \). Let \( A(G) \) be the adjacency matrix of \( G \) and \( \mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \) be the eigenvalues of \( A(G) \).

A good starting point that leads to Nikiforov’s spectral Erdős-Stone-Bollobás type theorem is the following theorem of Bollobás and Nikiforov.

**Theorem 1.3 (Bollobás, Nikiforov, [2])** For every graph \( G \) and integer \( p \geq 2 \)

\[
\mu^{p+1} \leq (p + 1)k_{p+1} + \sum_{s=2}^{p}(s - 1)k_s \mu^{p+1-s}
\]

In order to prove Theorem 1.3, we need a lemma whose proof does not rely on any spectral theory and is omitted.

**Lemma 1.4 (Nikiforov, [7])** If \( G \) is an \( n \)-vertex graph with \( 2 \leq p \leq \omega(G) \) then for every integer \( k \geq 1 \)
\[ \sum_{i=1}^{n} (k_p(v_i)w_{k+1}(v_i) - k_{p+1}(v_i)w_k(v_i)) \leq (p - 1)k_p w_k \]

**Proof of Theorem 1.3.** Assume that \( 2 \leq r < \omega(G) \). Fix an integer \( m \geq 1 \). Apply Lemma 1.4 with \( 2 \leq p \leq r, k = m + r - p \) and sum up these inequalities over \( p \) to obtain

\[ \sum_{i=1}^{n} k_2(v_i)w_{m+r-1}(v_i) - \sum_{i=1}^{n} k_r(v_i)w_m(v_i) \leq \sum_{p=2}^{r} (p - 1)k_p w_{m+r-p} \]

The leftmost term is just \( w_{m+r} \). Rearranging the inequality and using the fact that \( w_m(v_i) \leq w_{m-1} \) for every \( v_i \),

\[
\begin{align*}
w_{m+r} &\leq \sum_{i=1}^{n} k_r(v_i)w_m(v_i) + \sum_{p=2}^{r} (p - 1)k_p w_{m+r-p} \\
&\leq w_{m-1} \sum_{i=1}^{n} k_{r+1}(v_i) + \sum_{p=2}^{r} (p - 1)k_p w_{m+r-p} \\
&= w_{m-1}(r + 1)k_{r+1} + \sum_{p=2}^{r} (p - 1)k_p w_{m+r-p}
\end{align*}
\]

This implies

\[ \frac{w_{m+r}}{w_{m-1}} \leq (r + 1)k_{r+1} + \sum_{p=2}^{r} (p - 1)k_p \frac{w_{m+r-p}}{w_{m-1}} \]

There are nonnegative constants \( c_1, \ldots, c_n \) with \( c_1 > 0 \) such that

\[ w_m = c_1\mu_1^{m-1} + \cdots + c_n\mu_n^{m-1} \]

([4], page 44). Since \( \omega(G) > 2 \), \( G \) is not bipartite and so \( \mu_1 > |\mu_n| \) thus for every fixed integer \( s \geq 1 \)

\[ \lim_{m \to \infty} \frac{w_{m+s}}{w_{m-1}} = \mu^{s+1} \]

This equation together with the inequality proves Theorem 1.3 when \( 2 \leq r < \omega(G) \). The remaining cases are left to the reader.

See Exercise 3 for another example of a bound that is obtained using a similar technique.

Theorem 1.3 is then used to obtain a lower bound on \( k_{r+1} \) in terms of \( \mu \).

**Theorem 1.5 (Bollobás, Nikiforov, [2])** If \( G \) is an \( n \)-vertex graph and \( r \geq 2 \) then

\[ k_{r+1} \geq \left( \frac{n}{r} - 1 + \frac{1}{r} \right)^{r(r-1)\left( \frac{n}{r} \right)^{r+1}} \]
The idea of the proof of Theorem 1.5 is to first suppose there are many small cliques i.e., $k_s$ is big for some $2 \leq s \leq r$. A theorem of Moon and Moser (see [6], Ch. 10, Exercise 40) relates $k_{s+1}$ to $k_s$, so that if $k_s$ is big, then $k_{r+1}$ is big too. On the other hand, if $k_s$ is small for $2 \leq s \leq r$, then we apply Theorem 1.3 to get

$$(r + 1)k_{r+1} \geq \mu^{r+1} - \sum_{s=2}^{r} (s - 1)k_s \mu^{r+1-s}$$

and then substitute bounds on $k_s$ into the above inequality in order to obtain the desired bound on $k_{r+1}$.

Next Nikiforov uses Theorem 1.3 along with the following theorem whose proof does not use spectral theory.

**Theorem 1.6 (Nikiforov, [9])** Let $r \geq 2$ and $\epsilon > 0$. If $\epsilon^r \log n \geq 1$ and $G$ is an $n$-vertex graph with $k_r \geq \epsilon n^r$ then $K_r(s, \ldots, s, s') \subset G$ where $s = \lfloor \epsilon^r \log n \rfloor$ and $s' > n^{1-\epsilon^{-1}}$.

We conclude with Nikiforov’s spectral Erdös-Stone-Bollobás type theorem.

**Theorem 1.7 (Nikiforov, [10])** Let $r \geq 3$ be an integer and $c > 0$. If $(c/r^r) \log n \geq 1$ and $G$ is an $n$-vertex graph with

$$\mu(G) \geq (1 - \frac{1}{r-1} + c) n$$

then $K_r(s, \ldots, s, s') \subset G$ where $s \geq \lfloor (c/r^r) \log n \rfloor$ and $s' > n^{1-\epsilon^{-1}}$.

**Proof.** The hypothesis and Theorem 1.5 imply

$$k_r > \frac{c(r-1)(r-2)}{r} \left( \frac{n}{r-1} \right)^r > \frac{c}{r^r} n^r$$

Let $\epsilon = \frac{c}{r^r}$ and apply Theorem 1.6.

## 2 Exercises, Questions, and Problems

In all exercises $G$ is an $n$-vertex graph with $V(G) = \{v_1, \ldots, v_n\}$. A $k$-walk is a sequence of $k$ vertices $v_1, \ldots, v_k$ such that $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq k-1$ and $w_k$ is the number of $k$-walks in $G$. $A(G)$ is the adjacency matrix of $G$ and $\mu = \mu_1 \geq \cdots \geq \mu_n$ are its eigenvalues.

1. Show that for any $k$, there exists non-negative constants $c_1, \ldots, c_n$ with $c_1 > 0$ such that

$$w_k = \sum_{i=1}^{n} c_i \mu_i^{k-1}$$

2. Let $G$ be an $n$-vertex graph with $e$ edges and minimum degree $\delta$. In [7], Nikiforov proves the inequality
\[ \mu \leq \frac{\delta - 1}{2} + \sqrt{2e - n\delta + \frac{(1+\delta)^2}{4}} \]

Prove this inequality under the additional assumption that \( \mu > |\mu_n| \). Hint: The inequality is equivalent to

\[ \mu^2 - (\delta - 1)\mu - (2e - (n - 1)\delta) \leq 0 \]

For \( k \geq 4 \), note \( w_k = \sum_{i=1}^{n} w_{k-2}(v_i) \left( \sum_{v_j \in N(v_i)} d(v_j) \right) \). Derive the bound

\[ \sum_{v_j \in N(v_i)} d(v_j) \leq 2e - (n - 1 - d(v_i))\delta - d(v_i) \]

to obtain the inequality \( w_k \leq (2e - (n - 1)\delta)w_{k-2} + (\delta - 1)w_{k-1} \).

3. Question: If \( G \) is an \( n \)-vertex graph does \( \mu(G) < \mu(T_r(n)) \) imply \( e(G) < e(T_r(n)) \)?
   This question is due to Nikiforov [11].

4. Question: Let \( k_{r, t} \) be the number of \( K_r(t) \) subgraphs of \( G \). With this notation, Theorem 1.3 says that for any integer \( p \geq 2 \)

\[ \mu^{p+1} \leq (p + 1)k_{p+1, 1} + \sum_{s=2}^{p} (s - 1)k_{s, 1}\mu^{p+1-s} \]

Is there a similar formula for \( t \geq 2 \)?

5. Problem: Find a spectral version of the Erdős-Stone Theorem that relies entirely on spectral theory.

References


