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Eigenvalues of Random Power law Graphs

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Abstract. Many graphs arising in various information networks exhibit the "power law" behavior — the number of vertices of degree k is proportional to $k^{-\beta}$ for some positive β. We show that if $\beta > 2.5$, the largest eigenvalue of a random power law graph is almost surely $(1+o(1))\sqrt{m}$ where m is the maximum degree. Moreover, the k largest eigenvalues of a random power law graph with exponent β have power law distribution with exponent $2\beta-1$ if the maximum degree is sufficiently large, where k is a function depending on β, m and d, the average degree. When $2 < \beta < 2.5$, the largest eigenvalue is heavily concentrated at $cm^{3-\beta}$ for some constant c depending on β and the average degree. This result follows from a more general theorem which shows that the largest eigenvalue of a random graph with a given expected degree sequence is determined by m, the maximum degree, and $\tilde{d}$, the weighted average of the squares of the expected degrees. We show that the k-th largest eigenvalue is almost surely $(1+o(1))\sqrt{m_k}$ where m_k is the k-th largest expected degree provided m_k is large enough. These results have implications on the usage of spectral techniques in many areas related to pattern detection and information retrieval.

Keywords: random graphs, power law, eigenvalues

1. Introduction

Although graph theory has a history of more than 250 years, it is only very recently noted that the so-called "power law" is prevalent in realistic graphs arising in numerous arenas. Graphs with power law degree distribution are ubiquitous as observed in the Internet, the telecommunications graphs, email graphs and in various biological networks [2–4, 8, 13–15]. One of the basic problems concerns the distribution of the eigenvalues of power law graphs. In addition to theoretical interest, spectral methods are central in detecting clusters and finding patterns in various applications.

The eigenvalues of the adjacency matrices of various realistic power law graphs were computed and examined in [8, 9, 11]. Faloutsos et al. [8] conjectured a power law distribution for eigenvalues of power law graphs. For a fixed value $\beta > 1$, we say that a graph is a power law graph with exponent β if the number of vertices of

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degree k is proportional to $k^{-\beta}$. We note that for most realistic graphs, their power law models usually have exponents β falling between 2 and 3. For example, various Internet graphs [14] have exponents between 2.1 and 2.4. The Hollywood graph [4] has exponent $\beta \sim 2.3$. The telephone call graph [1] has exponent $\beta = 2.1$. Recently, Mihail and Papadimitriou [16] showed that the largest eigenvalue of a power law graph with exponent β has power law distribution if the exponent β of the power law graph satisfies $\beta > 3$.

In this paper, we will show that the largest eigenvalue λ of the adjacency matrix of a random power law graph is almost surely approximately the square root of the maximum degree m if $\beta > 2.5$, and the k largest eigenvalues of a random power law graph with exponent β have power law distribution with exponent $2\beta - 1$ if m is sufficiently large and k is small (to be specified later). When $2 < \beta < 2.5$, the largest eigenvalue of the adjacency matrix of a random power law graph is almost surely approximately $cm^{3-\beta}$. A phase transition occurs at $\beta = 2.5$. This result for power law graphs is an immediate consequence of a general result for eigenvalues of random graphs with arbitrary degree distribution.

We will use a random graph model from [5], which is a generalization of the Erdős-Rényi model, for random graphs with given expected degrees $w_1, w_2, \ldots, w_n$. The largest eigenvalue λ_1 of the adjacency matrix of a random graph in this model depends on two parameters — the maximum degree m and the second order average degree $\tilde{d}$ defined by

$$\tilde{d} = \frac{\sum_{i=1}^{n} w_i^2}{\sum_{i=1}^{n} w_i}.$$

It has turned out that λ_1 is almost surely $(1+o(1))\sqrt{m}$ if $\sqrt{m}$ is greater than $\tilde{d}$ by a factor of $\log^2 n$ and λ_1 is almost surely $(1+o(1))\tilde{d}$ if $\sqrt{m}$ is smaller than $\tilde{d}$ by a factor of $\log n$. In other words, λ is (asymptotically) the maximum of $\sqrt{m}$ and $\tilde{d}$ if the two values of $\sqrt{m}$ and d are far apart (by a power of $\log n$). Furthermore, If the k-th largest expected degree m_k is greater than $\tilde{d}$ by a factor of $\log n$, then the largest k eigenvalues are $(1+o(1))\sqrt{m_k}$.

One might be tempted to conjecture that

$$\lambda_1 = (1+o(1)) \max\{\sqrt{m}, \tilde{d}\}.$$

This, however, is not true as shown by a counter example in the last section. Throughout the paper, the asymptotic notation is used under the assumption that $n \to \infty$. We say that an event holds almost surely, if the probability that it holds tends to 1 as n tends to infinity.

Following the discussion in Mihail and Papadimitriou [16], our result has the following implications: The largest degree is a "local" aspect of a graph. If the largest eigenvalue depends only on the largest degree, spectral analysis of the Internet topology or spectral filtering for information retrieval can only be effective after high degree nodes have been normalized. Our result implies that such negative implications occur only when the exponent β exceeds 2.5.

2. Preliminaries

The primary model for classical random graphs is the Erdős-Rényi model $\mathcal{G}_p$, in which each edge is independently chosen with the probability p for some given $p > 0$ (see [7]). In such random graphs the degrees (the number of neighbors) of vertices all have the same expected value. Here we consider the following extended random graph model for a general degree distribution.

For a sequence $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, we consider random graphs $G(\mathbf{w})$ in which edges are independently assigned to each pair of vertices (i, j) with probability $w_i w_j \rho$, where $\rho = \frac{1}{\sum_{i=1}^n w_i}$. Notice that we allow loops in our model (for computational convenience) but their presence does not play any essential role. It is easy to verify that the expected degree of i is w_i.

To this end, we assume that $\max_i w_i^2 < \sum_k w_k$ so that $p_{ij} \le 1$ for all i and j. This assumption assures that the sequence w_i is graphical (in the sense that it satisfies the necessary and sufficient condition for a sequence to be realized by a graph [6]) except that we do not require the w_i's to be integers. We will use d_i to denote the actual degree of v_i in a random graph G in $G(\mathbf{w})$ where the weight w_i denotes the expected degree.

For a subset S of vertices, the volume $\mathrm{Vol}(S)$ is defined as the sum of weights in S. That is, $\mathrm{Vol}(S) = \sum_{i \in S} w_i$. In particular, we have $\mathrm{Vol}(G) = \sum_i w_i$, and we denote $\rho = \frac{1}{\mathrm{Vol}(G)}$. The induced subgraph on S is a random graph $G(\mathbf{w}')$ where the weight sequence is given by $w_i' = w_i \mathrm{Vol}(S)\rho$ for all $i \in S$. The second order average degree of $G(\mathbf{w}')$ is simply $\sum_{i \in S} w_i^2 \rho$.

The classical random graph $G(n, p)$ can be viewed as a special case of $G(\mathbf{w})$ by taking $\mathbf{w}$ to be $(pn, pn, \ldots, pn)$. In this special case, we have $\tilde{d} = d = m = np$. It is well known that the largest eigenvalue of the adjacency matrix of $G(n, p)$ is almost surely $(1 + o(1))np$ provided that $np \gg \log n$. Here we will determine the first eigenvalue of the adjacency matrix of a random graph in $G(\mathbf{w})$.

There are two easy lower bounds for the largest eigenvalues λ, namely, $(1 + o(1))\tilde{d}$ and $(1 + o(1))\sqrt{m}$. (The proofs can be found in Section 4.) Our main result states that the maximum of the above two lower bounds is essentially an upper bound.

Theorem 2.1. *For a graph G in $G(\mathbf{w})$, suppose the maximum degree m and the second order average degree $\tilde{d}$ satisfy $\tilde{d} > \sqrt{m}\log n$. Then the largest eigenvalue of G is almost surely $(1 + o(1))\tilde{d}$.*

Theorem 2.2. *For a graph G in $G(\mathbf{w})$, suppose the maximum degree m and the second order average degree $\tilde{d}$ satisfy $\sqrt{m} > \tilde{d}\log^2 n$. Then almost surely the largest eigenvalue of the adjacency matrix of G is $(1 + o(1))\sqrt{m}$.*

If the k-th largest expected degree m_k satisfies $\sqrt{m_k} > \tilde{d}\log^2 n$ and $m_k^2 \gg m\tilde{d}$, then almost surely the i-th largest eigenvalue of a random graph in $G(\mathbf{w})$ is $(1 + o(1))\sqrt{m_i}$, for all $1 \le i \le k$.

Theorem 2.3. *The largest eigenvalue of a random graph in $G(\mathbf{w})$ is at most $7\sqrt{\log n} \cdot \max\{\sqrt{m}, \tilde{d}\}$.*

We remark that with more careful analysis the factor of $\log n$ in Theorem 2.1 can be replaced by $(\log n)^{1/2+\varepsilon}$ and the factor of $\log^2 n$ can be replaced by $(\log n)^{3/2+\varepsilon}$ for any positive ε provided that n is sufficiently large. The constant "7" in Theorem 2.3 can be improved. We made no effort to get the best constant coefficient here.

As an application of Theorems 2.1 and 2.2, we prove that the largest eigenvalue of the random power law graph is $(1+o(1))\sqrt{m}$ if $\beta > 2.5$, and $(1+o(1))\tilde{d}$ if $\beta < 2.5$. A transition happens when $\beta = 2.5$.

3. Basic Facts

We will use the following concentration inequality for a sum of independent random variables (see [12]).

Lemma A. *Let X_i ($1 \le i \le n$) be independent random variables satisfying $|X_i| \le M$. Let $X = \sum_i X_i$. Then we have*

$$Pr(|X - E(X)| > a) \le e^{-\frac{a^2}{2(Var(X)+Ma/3)}}.$$

We will also use the following one-sided inequality [5]:

Lemma B. *Let $X_1, \ldots, X_n$ be independent random variables with*

$$Pr(X_i = 1) = p_i, \qquad Pr(X_i = 0) = 1 - p_i.$$

For $X = \sum_{i=1}^n a_i X_i$, we have $E(X) = \sum_{i=1}^n a_i p_i$, and we define $\nu = \sum_{i=1}^n a_i^2 p_i$. Then we have

$$Pr(X < E(X) - t) \le e^{-t^2/2\nu}.$$

The following lemma, due to Perron [17, p. 36], will also be very useful.

Lemma 3.1. *Suppose the entries of an $n \times n$ symmetric matrix A are all non-negative. For any positive constants $c_1, c_2, \ldots, c_n$, the largest eigenvalue $\lambda(A)$ satisfies*

$$\lambda(A) \le \max_{1 \le i \le n} \left\{ \frac{1}{c_i} \sum_{j=1}^n c_j a_{ij} \right\}.$$

Proof. Let C be the diagonal matrix $\mathrm{diag}(c_1, c_2, \ldots, c_n)$. Both A and $C^{-1}AC$ have the same eigenvalues. All entries of $C^{-1}AC$ are also non-negative. The first eigenvalue is bounded by the maximum row sum of $C^{-1}AC$. ∎

Now we are ready to state our key lemma.

Lemma 3.2. *For any given expected degree sequence $\mathbf{w}$, the largest eigenvalue λ_1 of a random graph in $G(\mathbf{w})$ is almost surely at most*

$$\tilde{d} + \sqrt{6\sqrt{m\log n}(\tilde{d} + \log n)} + 3\sqrt{m\log n}.$$

In particular, we have $\lambda_1 < 2\tilde{d} + 6\sqrt{m\log n}$.

Proof. For a fixed value x (to be chosen later), we define c_i, $1 \le i \le n$ as follows:

$$c_i = \begin{cases} w_i, & \text{if } w_i > x, \\ x, & \text{otherwise.} \end{cases}$$

Let A denote the adjacency matrix of G in $G(\mathbf{w})$. The entries a_{ij} of A are independent random variables. Now we apply Lemma 3.1, choosing $X_i = \frac{1}{c_i}\sum_{j=1}^{n} c_j a_{ij}$. We have

$$\begin{aligned} E(X_i) &= \frac{1}{c_i}\sum_{j=1}^{n} c_j w_i w_j \rho \\ &= \begin{cases} \sum_{w_j > x} w_j^2 \rho + x \sum_{w_j \le x} w_j \rho, & \text{if } w_i > x, \\ \frac{w_i}{x}\sum_{w_j > x} w_j^2 \rho + w_i \sum_{w_j \le x} w_j \rho, & \text{otherwise,} \end{cases} \\ &\le \tilde{d} + x; \end{aligned}$$

$$\begin{aligned} Var(X_i) &\le \frac{1}{c_i^2}\sum_{j=1}^{n} c_j^2 w_i w_j \rho \\ &= \begin{cases} \frac{1}{w_i}\sum_{w_j > x} w_j^3 \rho + \frac{x^2}{w_i}\sum_{w_j \le x} w_j \rho, & \text{if } w_i > x, \\ \frac{w_i}{x^2}\sum_{w_j > x} w_j^3 \rho + w_i \sum_{w_j \le x} w_j \rho, & \text{otherwise,} \end{cases} \\ &\le \frac{m}{x}\tilde{d} + x. \end{aligned}$$

By Lemma A, we have

$$Pr(|X_i - E(X_i)| > a) \le e^{-\frac{a^2}{2(Var(X_i) + ma/3x)}}.$$

Here we choose $x = \sqrt{m \log n}$ and $a = \sqrt{6(\frac{m}{x}\tilde{d} + x)\log n} + 2\frac{m}{x}\log n$. With probability at least $1 - o(\frac{1}{n})$, we have $X_i < \tilde{d} + x + a$ for every fixed $1 \le i \le n$. So we can conclude that almost surely $X_i < \tilde{d} + x + a$ holds simultaneously for all $1 \le i \le n$.

By Lemma 3.1, we have (almost surely)

$$\lambda \le \tilde{d} + \sqrt{6\sqrt{m \log n}(\tilde{d} + \log n)} + 3\sqrt{m \log n},$$

as desired. ∎

4. Proofs for the Main Theorems

This section presents the proofs of Theorems 2.1–2.3. We note that Theorem 2.1 is an easy consequence of Lemma 3.2. Theorem 2.2 requires some work and Theorem 2.3 is an immediate consequence of Lemma 3.2.

Proof of Theorem 2.1. We only need to prove the lower bound. Let A be the adjacency matrix of a random graph G in $G(\mathbf{w})$. We define

$$\alpha = \frac{1}{\sqrt{\sum_{i=1}^n w_i^2}}(w_1, w_2, \ldots, w_n)^*,$$

where $\mathbf{x}^*$ denotes the transpose of $\mathbf{x}$. Let

$$X = \alpha^* A \alpha = \frac{1}{\sum_{i=1}^n w_i^2}\left(2\sum_{i<j} w_i w_j X_{i,j} + \sum_i w_i^2 X_{i,i}\right).$$

Here $X_{i,j}$ is the 0-1 random variable with $Pr(X_{i,j} = 1) = w_i w_j \rho$. We will use Lemma B to prove a lower bound on X. Notice that

$$\begin{aligned} E(X) &= \frac{1}{\sum_{i=1}^n w_i^2}\left(2\sum_{i<j} w_i^2 w_j^2 \rho + \sum_i w_i^4 \rho\right) \\ &= \sum_{i=1}^n w_i^2 \rho \\ &= \tilde{d}, \end{aligned}$$

and

$$\begin{aligned} \nu &= \frac{1}{(\sum_{i=1}^n w_i^2)^2}\left(4\sum_{i<j} w_i^3 w_j^3 \rho + \sum_i w_i^6 \rho\right) \\ &\leq 2\left(\frac{\sum_{i=1}^n w_i^3}{\sum_{i=1}^n w_i^2}\right)^2 \rho \\ &\leq 2m^2\rho. \end{aligned}$$

Applying Lemma B with $t = \sqrt{2m^2\rho\log n}$, we have that with probability $1 - e^{-\log n/2} = 1 - o(1)$,

$$X > \tilde{d} - \sqrt{2m^2\rho\log n} = (1+o(1))\tilde{d}.$$

Since $\lambda \geq X$, it follows that almost surely $\lambda \geq (1+o(1))\tilde{d}$.

By the assumption of $\tilde{d} > \sqrt{m}\log n$, Lemma 3.2 implies that (almost surely) $\lambda \leq (1+o(1))\tilde{d}$. This and the previous fact complete the proof of Theorem 2.1. ∎

Proof of Theorem 2.2. We will first establish upper bounds for λ_i, $0 \leq 1 \leq k$, under the assumptions of Theorem 2.2. In the following proof, we use a weaker assumption that $\sqrt{m_k} > (\tilde{d}+1)\log^{1.5+\varepsilon} n$ for any positive ε. Note that $m = m_1$. We will first show that $\lambda_1 < (1+o(1))\sqrt{m}$.

Choose $s = \frac{m}{\log^{1+\varepsilon/2} n}$ and $t = \tilde{d}\log^{1+\varepsilon/2} n$. Let S denote the set of vertices with weight greater than s, and let T denote the set of vertices with weight less than or equal to t. Let $\overline{S}$ and $\overline{T}$ be the complements of S and T, respectively.

Since $s > t$, S and T are disjoint sets. G is covered by the following three subgraphs: $G(\overline{S})$ — the induced subgraph on $\overline{S}$, $G(\overline{T})$ — the induced subgraph on $\overline{T}$, and $G(S,T)$ — the bipartite graph between S and T. It is not hard to verify that $\mathrm{Vol}(S) \le \frac{\tilde{d}}{s\rho}$.

Both $G(\overline{S})$ and $G(\overline{T})$ are random graphs so that Lemma 3.2 can be applied. The maximum weight of $G(\overline{S})$ is at most s. We note that $\tilde{d}(G(\overline{S})) = \sum_{i \in \overline{S}} w_i^2 \rho \le \tilde{d}$. By Lemma 3.2, almost surely we have

$$\lambda_1(G(\overline{S})) \le 2\tilde{d} + 6\sqrt{s \log n} = o(\sqrt{m}).$$

Similarly $\tilde{d}(G(\overline{T})) = \sum_{i \in \overline{T}} w_i^2 \rho \le \tilde{d}$. The maximum weight of $G(\overline{T})$ is at most

$$m\mathrm{Vol}(\overline{T})\rho \le m\frac{\tilde{d}}{t} = \frac{m}{\log^{1+\varepsilon/2} n}.$$

By Lemma 3.2, almost surely we have

$$\lambda_1(G(\overline{T})) \le 2\tilde{d} + 6\sqrt{\frac{m}{\log^{1+\varepsilon/2} n}\log n} = o(\sqrt{m}).$$

Next we consider the largest eigenvalue of $G(S,T)$.

Claim 1. The following holds almost surely. For any vertex $i \in S$, all but $\tilde{d}^2 \log^{2+\varepsilon} n$ of its neighbors in T have degree 1 in $G(S,T)$.

Proof of Claim 1. Fix a vertex $i \in S$ and expose its neighbors in T. With probability $1 - o(1/n)$, i has at most $(1+o(1))m$ neighbors in T. For any neighbor $k \in T$ of i, the expected number of neighbors of k (other than i) in S is at most

$$\mu \le E\left(\sum_{j \in S\setminus i} X_{kj}\right) \le w_k \mathrm{Vol}(S)\rho \le t\frac{\tilde{d}}{s} = \frac{\tilde{d}^2}{m}\log^{2+\varepsilon} n < \frac{1}{\log n}.$$

It follows that the expected number of neighbors of i with more than one neighbors in S is at most $(1+o(1))m\mu \le (1+o(1))\tilde{d}^2 \log^{2+\varepsilon} n$. Using Lemma A, it is easy to show that with probability $1-o(1/n)$, the number of neighbors of i with more than one neighbors in S is at most $(1+o(1))\tilde{d}^2 \log^{2+\varepsilon} n$. The claim follows from the union bound.

Claim 2. Almost surely, the maximum degree of vertices in T in $G(S,T)$ is at most $3\log n$.

Proof of Claim 2. The expected degree of a vertex $i \in T$ in $G(S,T)$ is $w_i \mathrm{Vol}(S)\rho \le \frac{t\tilde{d}}{s} < \frac{1}{\log n}$. A routine application of Lemma A shows (with room to spare) that with probability $1-o(1/n)$, the degree of i in S is at most $3\log n$. Again the union bound completes the proof.

Let G_1 be the subgraph of $G(S,T)$ consisting of all edges with degree 1 in T. Let G_2 be the subgraph of $G(S,T)$ consisting of all edges not in G_1. G_1 is a disjoint union of stars. The maximum expected degree is at most $(1+o(1))m$. We have

$$\lambda_1(G_1) \le (1+o(1))\sqrt{m}.$$

The largest eigenvalue of G_2 is bounded above by $\sqrt{m_S m_T}$, where m_S and m_T are the maximum degrees in S and T, respectively. Claims 1 and 2 show that $m_S \leq \tilde{d}^2 \log^{2+\varepsilon} n$ and $m_T \leq \log n$. By Lemma 3.1, we have

$$\lambda_1(G_2) \leq \sqrt{m_S m_T} \leq \tilde{d} \log^{3/2+\varepsilon/2} n = o(\sqrt{m}).$$

Hence, we have

$$\lambda_1(G) \leq \lambda_1(G(\overline{S})) + \lambda_1(G(\overline{T})) + \lambda_1(G_1) + \lambda_1(G_2) \leq (1+o(1))\sqrt{m}.$$

Now, consider $G' = G \setminus \{v\}$ for any vertex v. Let $\lambda_i(G)$ denote the i-th largest eigenvalue of G. The well-known interlacing theorem (see [10]) asserts that

$$\lambda_i(G) \geq \lambda_i(G') \geq \lambda_{i+1}(G).$$

Suppose that vertex v_i has the i-th largest expected degree m_i and $G_i = G \setminus \{v_1, \ldots, v_{i-1}\}$. It is easy to check that the second order average degree of G_i is not greater than $\tilde{d}$ and the largest expected degree of G_i is $(1+o(1))m_i$ provided $i \leq k$. By the first part of the theorem, we have

$$\lambda_1(G_i) \leq (1+o(1))\sqrt{m_i}.$$

By repeated using interlacing theorem, we have

$$\lambda_i(G) = \lambda_i(G_1) \leq \lambda_{i-1}(G_2) \leq \ldots \leq \lambda_1(G_i) \leq (1+o(1))\sqrt{m_i}.$$

Now we turn to the lower bound on λ_i's. We will use two helpful facts that are immediate consequences of the interlacing theorem and the Courant-Fisher theorem.

Claim 3. Suppose H is an *induced* subgraph of G. Then $\lambda_i(G) \geq \lambda_i(H)$ for all $1 \leq i \leq |V(H)|$.

Claim 4. Suppose F is a subgraph of a graph H. Then

$$\lambda_i(H) \geq \lambda_i(F) - \lambda_1(F'),$$

where F' has edge set consisting of all edges of H not in F.

To prove the lower bound $\lambda_i > (1+o(1))\sqrt{m_i}$, it suffices to find an induced subgraph H of G with eigenvalues $\lambda_i(H) \geq 1+o(1)\sqrt{m_i}$ for $1 \leq i \leq k$. Let S consist of vertices with weights $m_1, \ldots, m_k$. Let U denote the set of neighbors of S in T where T is defined as before. Let H be the induced subgraph of G on $S \cup U$. H is the union of three graphs: the induced graphs $G(S)$, $G(U)$, and the bipartite graph $G(S,U)$.

$G(T)$ is a random graph and $G(U)$ is a subgraph of $G(T)$. By Lemma 3.2, we have

$$\lambda_1(G(U)) \leq \lambda_1(G(T)) \leq 2\tilde{d} + 6\sqrt{t \log n} = o(\sqrt{m_k}).$$

The maximum weight of $G(S)$ is at most $m\mathrm{Vol}(S)\rho$. We consider two possibilities:

Case 1. $m\mathrm{Vol}(S)\rho < \overline{d} \log^2 n$. In this case, we have

$$\lambda_1(G(S)) \leq 2\tilde{d} + 6\sqrt{m\mathrm{Vol}(S)\rho \log n} = o(\sqrt{m_k}).$$

Case 2. $m\mathrm{Vol}(S)\rho > \overline{d}\log^2 n$. In this case, we have

$$\lambda_1(G(S)) \leq (1+o(1))\sqrt{m\mathrm{Vol}(S)\rho} \leq (1+o(1))\sqrt{\frac{m\tilde{d}}{m_k}} \leq o(\sqrt{m_k}),$$

since $m_k^2 \gg m\tilde{d}$. In both cases, we use the inequality

$$\mathrm{Vol}(S)\min_{i\in S} w_i \leq \tilde{d}\mathrm{Vol}(G),$$

which follows easily from the definitions of Vol and $\tilde{d}$.

In the bipartite graph $G(S,U)$, we define a spanning forest F as follows. The edges of F are, for $i = 1,\ldots,k$, from vertex i to $U \setminus \cup_{j=1}^{i-1}(\Gamma(j)\cap T)$ where $\Gamma(j)\cap T$ is the neighbors of j in T. Let R be the bipartite subgraph containing edges not in F.

The volume of T is almost equal to the volume of G since

$$\mathrm{Vol}(T) = \mathrm{Vol}(G) - \mathrm{Vol}(\overline{T}) \geq \mathrm{Vol}(G)\left(1-\frac{\tilde{d}}{t}\right) = (1-o(1))\mathrm{Vol}(G).$$

Thus, the size of $\Gamma(j)\cap T$ almost surely is $(1+o(1))m_j$. We have

$$\mathrm{Vol}(\cup_{j=1}^{i-1}\Gamma(j)\cap T) \leq \sum_{j=1}^{i-1}(1+o(1))m_j t \leq (1+o(1))\frac{\tilde{d}}{m_i}t\mathrm{Vol}(G).$$

The expected degree of i in R is at most

$$m_i\mathrm{Vol}(\cup_{j=1}^{i-1}\Gamma(j)\cap T)\rho \leq (1+o(1))\tilde{d}t.$$

By Chernoff's inequalities, it is easy to show that the maximum degree of i in R is at most $2\tilde{d}t$. On the other hand, the maximal degree of any vertex $u\in U$ in R is at most $3\log n$ by Claim 2. Therefore, we have

$$\lambda_1(R) \leq \sqrt{2\tilde{d}t3\log n} = o(\sqrt{m_k}).$$

Now, F is the disjoint union of k stars with sizes $(1+o(1))m_i$ for $i=1,\ldots,k$. We have $\lambda_i(F) = (1+o(1))\sqrt{m_i}$, for $i=1,\ldots,k$. Hence, we have

$$\lambda_i(G) \geq \lambda_i(H) \geq \lambda_i(F) - \lambda_1(G(S)) - \lambda_1(G(U)) - \lambda_1(R) = (1+o(1))\sqrt{m_i},$$

for $1 \leq i \leq k$, completing the proof. ∎

5. Random Power law Graphs

In this section, we consider random graphs with power law degree distribution with exponent β. We choose the degree sequence $G(\mathbf{w}) = (w_1, w_2, \ldots, w_n)$ satisfying $w_i = ci^{-\frac{1}{\beta-1}}$ for $i_0 \leq i \leq n+i_0$. Here c is determined by the average degree and i_0 depends on the maximum degree m, namely, $c = \frac{\beta-2}{\beta-1} d n^{\frac{1}{\beta-1}}$, $i_0 = n\left(\frac{d(\beta-2)}{m(\beta-1)}\right)^{\beta-1}$. It is easy to verify that the number of vertices of degree k is proportional to $k^{-\beta}$.

The second order average degree $\tilde{d}$ can be computed as follows:

$$\tilde{d} = \begin{cases} d\frac{(\beta-2)^2}{(\beta-1)(\beta-3)}(1+o(1)), & \text{if } \beta > 3, \\ \frac{1}{2} d \ln \frac{2m}{d}(1+o(1)), & \text{if } \beta = 3, \\ d\frac{(\beta-2)^2}{(\beta-1)(3-\beta)}\left(\frac{(\beta-1)m}{d(\beta-2)}\right)^{3-\beta}(1+o(1)), & \text{if } 2 < \beta < 3. \end{cases}$$

We remark that for $\beta > 3$, the second order average degree is independent of the maximum degree. Consequently, the power law graphs with $\beta > 3$ are much easier to deal with. However, many massive graphs are power law graphs with $2 < \beta < 3$, in particular, the Internet graphs [14] have exponents between 2.1 and 2.4.

Theorem 5.1. *We have*

(1) *For* $\beta \geq 3$*, suppose the maximum degree* m *satisfies*

$$m > d^2 \log^3 n, \tag{5.1}$$

where d *is the average degree. Then almost surely the largest eigenvalue of the random power law graph* G *is* $(1+o(1))\sqrt{m}$.

(2) *For* $3 > \beta > 2.5$*, suppose* m *satisfies*

$$m > d^{\frac{\beta-2}{\beta-2.5}} \log^{\frac{3}{\beta-2.5}} n. \tag{5.2}$$

Then almost surely the largest eigenvalue of the random power law graph G *is* $(1+o(1))\sqrt{m}$.

(3) *For* $2 < \beta < 2.5$ *and* $m > \log^{\frac{3}{2.5-\beta}} n$*, almost surely the largest eigenvalue is* $(1+o(1))\tilde{d}$.

(4) *For* $k < n(\frac{d}{m\log n})^{\beta-1}$ *and* $\beta > 2.5$*, almost surely the* k *largest eigenvalues of the random power law graph* G *with exponent* β *have power law distribution with exponent* $2\beta-1$*, provided that* m *is large enough* (*satisfying* (5.1), (5.2)).

We remark that the powers of $\log n$ can be slightly improved, as well as the results for the case of $\beta = 3$. We do not attempt to optimize such estimates here.

Proof of Theorem 5.1. If $\beta \geq 3$, clearly $\sqrt{m} > \tilde{d} \log^{2/3} n$. By Theorem 2.2, almost surely the largest eigenvalue of the random power law graph G is $(1+o(1))\sqrt{m}$.

If $3 > \beta > 2.5$, it is straightforward to verify that $\sqrt{m} > \tilde{d} \log^3 n$. By Theorem 2.2, almost surely the largest eigenvalue of the random power law graph G is $(1+o(1))\sqrt{m}$.

When $\beta < 2.5$, we have $\tilde{d} > \sqrt{m} \log n$. The result follows from Theorem 2.1.

To prove (4), we first note that $k \leq n(\frac{d}{m\log n})^{\beta-1}$ implies that $k \leq n/(d\log^4 n)^{\beta-1}$ for $\beta \geq 3$, and $k < n/(d\log^7 n)^{(\beta-1)/(2\beta-5)}$ for $3 > \beta > 2.5$.

Now, for $k \le n/(d\log^4 n)^{\beta-1}$ and $\beta \ge 3$, we have $k < i_0[(\frac{m}{d^2\log^3 n})^{\beta-1} - 1]$. Thus,

$$m_k = m\left(\frac{i_0}{i_0+k}\right)^{1/(\beta-1)} \ge d^2\log^3 n.$$

For $3 > \beta > 2.5$ and $k < n/(d\log^7 n)^{(\beta-1)/(2\beta-5)}$, we have

$$k < i_0\left[\left(\frac{m}{d^{\beta-2)/(\beta-2.5)}\log^{3/(\beta-2.5)} n}\right)^{\beta-1} - 1\right].$$

Thus,

$$m_k = m\left(\frac{i_0}{i_0+k}\right)^{1/(\beta-1)} > d^{\frac{\beta-2}{\beta-2.5}}\log^{\frac{3}{\beta-2.5}} n.$$

In both cases, one can verify that the assumptions of Theorem 2.2 are met. Thus, Theorem 2.2 implies that for all $1 \le i \le k$, the i-th largest eigenvalue is (almost surely) $(1+o(1))\sqrt{m_i}$. On the other hand, the m_i's have a power distribution with exponent β. By a routine calculation, one can show that $\sqrt{m_i}$'s have a power distribution with exponent $2\beta - 1$ and this concludes the proof. ∎

6. Problems and Remarks

We have proved that the largest eigenvalue λ of G in $G(\mathbf{w})$ is roughly equal to $\tilde{d}$ or $\sqrt{m}$ if one of them is much larger than the other. What happens when $\tilde{d}$ and $\sqrt{m}$ are comparable? Is it true that $\lambda = (1+o(1))\max\{\sqrt{m}, \tilde{d}\}$? The following example shows that $\lambda(G)$ can be larger than $\tilde{d}$ and $\sqrt{m}$ by a constant factor.

Example 6.1. For given m satisfying $m > \log^2 n$ and d constant, we choose the expected degree sequence as follows: There are $n_1 = \frac{nd}{m^{3/2}} = o(n)$ vertices with weight m. The remaining vertices have weight d. We then have

$$\mathrm{Vol}(G) = n_1 m + (n-n_1)d \approx nd,$$

$$\tilde{d} = n_1 m^2\rho + (n-n_1)d^2\rho \approx \sqrt{m}.$$

Our random graph is defined with this special degree sequence.

Claim 5. The largest eigenvalue λ of the adjacency matrix of $G(\mathbf{w})$ is almost surely at least $(1-o(1))\frac{1+\sqrt{5}}{2}\sqrt{m} > 1.618\sqrt{m}$.

Proof of Claim 5. Let S be the set of vertices with weight m, and T be the remaining vertices. Since $\mathrm{Vol}(S) \approx \frac{1}{\sqrt{m}}\mathrm{Vol}(G)$ and $\mathrm{Vol}(T) \approx \mathrm{Vol}(G)$, the expected number of neighbors in T for a vertex in S is about m, while the expected number of neighbors in S for a vertex in T is about $\frac{d}{\sqrt{m}} = o(1)$. It can be shown that a random graph G in $G(\mathbf{w})$ almost surely contains n_1 disjoint union of stars of size $m' = (1+o(1))m$ with centers in S. As always, A denotes the adjacency matrix of G.

Recall that $\lambda(A) \geq \alpha^* A \alpha$ for any unit vector α. We next present a vector α such that the expectation of $\alpha^* A \alpha$ is significantly larger than $\sqrt{m}$. For any vertex u, the coordinates α_u is defined as follows:

$$\alpha_u = \begin{cases} \frac{\sqrt{c}}{\sqrt{n_1}}, & \text{if } u \in S, \\ \frac{\sqrt{1-c}}{\sqrt{n_1 m'}}, & \text{if } u \text{ is a leaf of the stars,} \\ 0, & \text{otherwise.} \end{cases}$$

Here $c = (1+1/\sqrt{5})/2$ is a constant maximizing $E(\alpha^* A \alpha)$.

Clearly, α is a unit vector. We have

$$\begin{aligned} E(\alpha^* A \alpha) &\geq n_1^2 m^2 \rho \frac{c}{n_1} + 2 n_1 m' \frac{\sqrt{c}}{\sqrt{n_1}} \frac{\sqrt{1-c}}{\sqrt{n_1 m'}} \\ &\approx \sqrt{m}(c + 2\sqrt{c(1-c)}) \\ &= \frac{1+\sqrt{5}}{2}\sqrt{m}. \end{aligned}$$

With the assumption $m > \log^2 n$, we conclude that almost surely $\alpha^* A \alpha$ is greater than $(\frac{1+\sqrt{5}}{2} + o(1))\sqrt{m}$, completing the proof. ∎

We proved that the statement $\lambda = (1+o(1))\max\{\sqrt{m}, \tilde{d}\}$ is false. However, it looks plausible that λ could be (almost surely) upper bounded by $(1+o(1))(\tilde{d} + \sqrt{m})$ provided that $\tilde{d} + \sqrt{m}$ is sufficiently large (i.e., $\omega(\log n)$).

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