A generalized Alon-Boppana bound and weak Ramanujan graphs

Fan Chung *

Abstract

A basic eigenvalue bound due to Alon and Boppana holds only for regular graphs. In this paper we give a generalized Alon-Boppana bound for eigenvalues of graphs that are not required to be regular. We show that a graph $G$ with diameter $k$ and vertex set $V$, the smallest nontrivial eigenvalue $\lambda_1$ of the normalized Laplacian $L$ satisfies

$$\lambda_1 \leq 1 - \sigma(1 - \frac{c}{k})$$

for some constant $c$ where $\sigma = 2 \sum_v d_v \sqrt{d_v - 1}/\sum_v d_v^2$ and $d_v$ denotes the degree of the vertex $v$.

We consider weak Ramanujan graphs defined as graphs satisfying $\lambda_1 \geq 1 - \sigma$. We examine the vertex expansion and edge expansion of weak Ramanujan graphs and then use the expansion properties among other methods to derive the above Alon-Boppana bound.

1 Introduction

The well-known Alon-Boppana bound [8] states that for any $d$-regular graph with diameter $k$, the second largest eigenvalue $\rho$ of the adjacency matrix satisfies

$$\rho \geq 2\sqrt{d-1} \left( 1 - \frac{2}{k} \right) - \frac{2}{k}.$$  (1)

A natural question is to extend Alon-Boppana bounds for graphs that are irregular. Hoory [6] showed that for an irregular graph, the second largest eigenvalue $\rho$ of the adjacency matrix satisfies

$$\rho \geq 2\sqrt{d-1} \left( 1 - \frac{c \log r}{r} \right)$$

*University of California, San Diego. Research supported in part by AFSOR FA9550-09-1-0090
if the average degree of the graph after deleting a ball of radius $r$ is at least $d$ where $r, d > 2$.

For irregular graphs, it is often advantageous to consider eigenvalues of the normalized Laplacian for deriving various graph properties. For a graph $G$, the normalized Laplacian $\mathcal{L}$, defined by

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2}$$

where $D$ is the diagonal degree matrix and $A$ denotes the adjacency matrix of $G$. One of the main tools for dealing with general graphs is the Cheeger inequality which relates the least nontrivial eigenvalue $\lambda_1$ to the Cheeger constant $h_G$:

$$2h_G \geq \lambda_1 \geq \frac{h_G^2}{2}$$

where $h_G = \min_S |\partial(S)|/\text{vol}(S)$ for $S$ ranging over all vertex subsets with volume $\text{vol}(S) = \sum_{u \in S} d_u$ no more than half of $\sum_{u \in V} d_u$ and $\partial(S)$ denotes the set of edges leaving $S$. For $k$-regular graphs, we have $\lambda_1 = 1 - \rho/k$ where $\rho$ denotes the second largest eigenvalue of the adjacency matrix. In general,

$$\frac{\rho}{\max_v d_v} \leq 1 - \lambda_1 \leq \frac{\rho}{\min_v d_v}$$

which can be used to derive a version of the Cheeger inequality involving $\rho$ which is less effective than (2) for irregular graphs.

In this paper, we will show that for a connected graph $G$ with diameter $k$, $\lambda_1$ is upper bounded by

$$\lambda_1 \leq 1 - \sigma (1 - \frac{c}{k})$$

for a constant $c$ where $\sigma = 2 \sum_v d_v \sqrt{d_v} - 1/\sum_v d_v^2$. The above inequality will be proved in Section 6.

The above bound of Alon-Boppana type improves a result of Young [10] who derived a similar eigenvalue bound using a different method. In [10] the notion of $(r, d, \delta)$-robust graphs was considered and it was shown that for a $(r, d, \delta)$-robust graph, the least nontrivial eigenvalue $\lambda_1$ satisfies

$$\lambda_1 \leq 1 - \frac{2d\sqrt{d - 1}}{\delta} \left(1 - \frac{c}{r}\right).$$

Here $(r, d, \delta)$-robustness means for every vertex $v$ and the ball $B_r(v)$ consisting of all vertices with distance at most $r$, the induced subgraph on the complement of $B_r(v)$ has average degree at least $d$ and $\sum_{v \notin B_r(v)} d_v^2 / |V \setminus B_r(v)| \leq \ldots$
δ. We remark that our result in (3) does not require the condition of robustness.

We define weak Ramanujan graphs to be graphs with eigenvalue \( \lambda_1 \) satisfying

\[
\lambda_1 \geq 1 - \sigma \geq \frac{1}{2}
\]

where \( \sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2 \).

To prove the Alon-Boppana bound in (3), it suffices to consider only weak Ramanujan graphs. Weak Ramanujan graphs satisfy various expansion properties. We will describe several vertex-expansion and edge-expansion properties involving \( \lambda_1 \) in Section 3, which will be needed later for proving a diameter bound for weak Ramanujan graphs in Section 4. The diameter bound and related properties of weak Ramanujan graphs are useful in the proof of the Alon-Boppana bound for general graphs.

We will also show that the largest eigenvalue \( \lambda_{n-1} \) of the normalized Laplacian satisfies

\[
\lambda_{n-1} \geq 1 + \sigma (1 - \frac{c}{k}).
\]

The proof will be given in Section 7.

\section{Preliminaries}

For a graph \( G = (V, E) \), we consider the normalized Laplacian

\[
\mathcal{L} = I - D^{-1/2} AD^{-1/2}
\]

where \( A \) denotes the adjacency matrix and \( D \) denotes the diagonal degree matrix with \( D(v, v) = d_v \), the degree of \( v \). We assume that there is no isolated vertex throughout this paper. For a vertex \( v \) and a positive integer \( l \), let \( B_l(v) \) denote the ball consisting of all vertices within distance \( l \) from \( v \). For an edge \( \{x, y\} \in E \) we say \( x \) is adjacent to \( y \) and write \( x \sim y \).

Let \( \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1} \) denote eigenvalues of \( \mathcal{L} \), where \( n \) denotes the number of vertices in \( G \). It can be checked (see [2]) that \( \lambda_1 > 0 \) if \( G \) is connected. The Alon-Boppana bound obviously holds if \( \lambda_1 = 0 \). In the remainder of this paper, we assume \( G \) is connected.

Let \( \varphi_i \) denote the orthonormal eigenvector associated with eigenvalue \( \lambda_i \). In particular, \( \varphi_0 = D^{1/2} 1 / \sqrt{\text{vol}(G)} \) where \( 1 \) is the all 1’s vector and
vol(G) = \sum_{v \in V} d_v. We can then write

\[
\lambda_1 = \inf_{g \perp \varphi_0} \frac{\langle g, Lg \rangle}{\langle g, g \rangle} = \inf_{f \perp D1} \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_z f(z)^2} = \inf_{f \perp D1} R(f)
\]

where \(f\) ranges over all functions satisfying \(\sum_u f(u) d_u = 0\) and the sum \(\sum_{x \sim y}\) ranges over all unordered pairs \(\{x, y\}\) where \(x\) is adjacent to \(y\). Here \(R(f)\) denote the Rayleigh quotient of \(f\), which can be written as follows:

\[
R(f) = \frac{\int |\nabla f|}{\int \|f\|^2}
\]

where \(\int \|f\|^2 = \sum_x f^2(x) d_x\) and \(\int |\nabla f| = \sum_{x \sim y} (f(x) - f(y))^2\).

For eigenfunction \(\varphi_i\), the function \(f_i = D^{-1/2}\varphi_i\), called the combinatorial eigenfunction associated with \(\lambda_i\), satisfies

\[
\lambda_i f(u) d_u = \sum_{v \sim u} (f(u) - f(v))
\]

for each vertex \(u\). In particular, for \(f\) satisfying \(\sum_u f(u) d_u = 0\), we have

\[
\langle f, Af \rangle \leq (1 - \lambda_1) \langle f, Df \rangle
\]

and

\[
|\langle f, Af \rangle| \leq \max_{i \neq 0} (1 - \lambda_i) \langle f, Df \rangle.
\]

3 Vertex and edge expansions

For any subset \(S\) of vertices, there are two types of boundaries. The edge boundary of \(S\), denoted by \(\partial(S)\) consists of all edges with exactly one endpoint in \(S\). The vertex boundary of \(S\), denoted by \(\delta(S)\) consists of all vertices not in \(S\) but adjacent to vertices in \(S\). Namely,

\[
\partial(S) = \{\{u, v\} \in E : u \in S \text{ and } v \notin S\} = E(S, \bar{S})
\]

\[
\delta(S) = \{u \notin S : u \sim v \in S \text{ for some vertex } v\}
\]
In this section, we will examine vertex expansion and edge expansion relying only on $\lambda_1$. These expansion properties will be needed for deriving diameter bounds for weak Ramanujan graphs which will be used in our proof of the general Alon-Boppana bound later in Section 6.

From the definition of the Cheeger constant, for all vertex subsets $S$, we have

$$\frac{\left|\partial(S)\right|}{\text{vol}(S)} \geq h_G \geq \frac{\lambda_1}{2}$$

Later in the proofs, we will be interested in the case that $\text{vol}(S)$ is small and therefore we will use the following version.

**Lemma 1** Let $S$ be a subset of vertices in $G$. Then

$$\frac{\left|\partial(S)\right|}{\text{vol}(S)} \geq \lambda_1 \left(1 - \frac{\text{vol}(S)}{\text{vol}(G)}\right).$$

**Proof:** Suppose $f$ is defined by

$$f = \frac{1_S}{\text{vol}(S)} - \frac{1_{\bar{S}}}{\text{vol}(S)}$$

where $1_S$ denotes the characteristic function defined by $1_S(v) = 1$ if $v \in S$ and 0 otherwise.

The Rayleigh quotient $R(f)$ satisfies

$$\lambda_1 \leq R(f) = \frac{\left|\partial(S)\right|}{\text{vol}(S)} \cdot \frac{\text{vol}(G)}{\text{vol}(S)}.$$ 

$$\square$$

For the expansion of the vertex boundary, the Tanner bound [9] for regular graphs can be generalized as follows.

**Lemma 2** Let $\bar{\lambda} = \min_{i \neq 0} |1 - \lambda_i|$. Then for any vertex subset $S$ in a graph,

$$\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{1 - \bar{\lambda}^2}{\bar{\lambda}^2 + \frac{\text{vol}(S)}{\text{vol}(G)}} \quad (10)$$

The proof of the above inequality is by using the following discrepancy inequality (as seen in [2]).
Lemma 3 In a graph $G$, for two subset $X$ and $Y$ of vertices, the number $e(X, Y) = |E(X, Y)|$ of edges between $X$ and $Y$ satisfies

$$
\left| e(X, Y) - \frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)} \right| \leq \overline{\lambda} \sqrt{\frac{\text{vol}(X)\text{vol}(Y)\text{vol}(X)\text{vol}(Y)}{\text{vol}(G)^2}}
$$

(11)

where $\overline{\lambda} = \min_{i \neq 0} |1 - \lambda_i|$.

The proof of Lemma 3 follows from (9) and can be found in [2]. The proof of (12) results from (11) by setting $X = S$ and $Y = S \cup \delta(S)$.

Here we will give a version of the vertex-expansion bounds for general graphs which only rely on $\lambda_1$ and are independent of other eigenvalues.

Lemma 4 In a graph $G$ with vertex set $V$ and the first nontrivial eigenvalue $\lambda_1$, for a subset $S$ of $V$ with $\text{vol}(S \cup \delta S) \leq \epsilon \text{vol}(G) \leq \text{vol}(G)/2$, the vertex boundary of $S$ satisfies

(i) \[ \frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{2\lambda_1}{1 - \lambda_1 + 2\epsilon} \] (12)

(ii) If $1/2 \leq \lambda_1 \leq 1 - 2\epsilon$, then\[ \frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{1}{(1 - \lambda_1 + 2\epsilon)^2}. \] (13)

Proof: The proof of (i) follows from Lemma 1 since

\[
\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{|\partial(S \cup \delta(S))| + |\partial(S)|}{\text{vol}(S)} \\
\geq \frac{\lambda_1(1 - \epsilon)(\text{vol}(S) + \text{vol}(\delta(S))) + \lambda_1(1 - \epsilon)\text{vol}(S)}{\text{vol}(S)}
\]

Therefore

\[
\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{2\lambda_1(1 - \epsilon)}{1 - \lambda_1(1 - \epsilon)} \geq \frac{2\lambda_1}{1 - \lambda_1 + 2\epsilon}
\]

To prove (ii), we set $f = 1_S + \gamma 1_{\delta(S)}$ where $\gamma = 1 - \lambda_1$. Consider $g = f - c 1_V$ where $c = \sum_u f(u)d_u/\text{vol}(G)$. By the Cauchy-Schwarz inequality, we have

\[
c^2 = \frac{1}{(\text{vol}(G))^2} \left( \sum_{u \in S \cup \delta(S)} f(u)d_u \right)^2 \leq \frac{\text{vol}(S \cup \delta(S))}{(\text{vol}(G))^2} \sum_u f^2(u)d_u \\
\leq \frac{\epsilon}{\text{vol}(G)} \sum_u f^2(u)d_u.
\]
Using the inequality in (8), we have
\[
\langle f, Af \rangle \leq \langle g, Ag \rangle + c^2 \text{vol}(G) \\
\leq \gamma \langle f, Dg \rangle + c^2 \text{vol}(G) \\
= \gamma \langle f, Df \rangle + (1 - \gamma)c^2 \text{vol}(G) \\
\leq (\gamma + \epsilon)\langle f, Df \rangle \\
= (\gamma + \epsilon)(\text{vol}(S) + \gamma^2 \text{vol}(\delta(S))).
\]

Let \( e(S, T) \) denote the number of ordered pairs \((u, v)\) where \( u \in S, v \in T \) and \( \{u, v\} \in E \). Since \( \gamma = 1 - \lambda \leq 1/2 \), we have
\[
\langle f, Af \rangle \geq e(S, S) + 2\gamma e(S, \delta(S)) \\
\geq (1 - 2\gamma)e(S, S) + 2\gamma \text{vol}(S) \\
\geq 2\gamma \text{vol}(S)
\]
Together we have
\[
\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{\gamma - \epsilon}{\sigma^2(\gamma + \epsilon)} \\
\geq \frac{1}{(\gamma + 2\epsilon)^2}
\]
since \( \gamma \geq 2\epsilon \). \( \Box \)

Recall that weak Ramanujan graphs have eigenvalue \( \lambda_1 \) satisfying
\[
\lambda_1 \geq 1 - \sigma
\]
where \( \sigma = 2 \sum_v d_v \sqrt{d_v - 1}/\sum_v d_v^2 \). Lemma 1 implies that for \( S \) with \( \text{vol}(S \cup \delta(S)) \leq \epsilon \text{vol}(G) \),
\[
\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{1}{(\sigma + 2\epsilon)^2}.
\]
For \( k \)-regular Ramanujan graphs with eigenvalue \( \lambda_1 = 1 - 2\sqrt{k - 1}/k \), the above inequality is consistent with the bound
\[
\frac{\text{vol}(\delta(S))}{\text{vol}(S)} = \frac{\lvert \delta(S) \rvert}{\lvert S \rvert} \geq \frac{1}{(2\sqrt{k - 1}/k + 2\epsilon)^2}
\]
which is about \( k/4 \) when \( \text{vol}(S) \) is small. The factor \( k/4 \) in the above inequality was improved by Kahale [4] to \( k/2 \). There are many applications
(see [1]) that require graphs having expansion factor to be \((1 - \epsilon)k\). Such graphs are called lossless expanders. In [1], lossless graphs were constructed explicitly by using the zig-zag construction but the method for deriving the expansion bounds does not use eigenvalues. In this paper, the expansion factor as in Lemma 4 is enough for our proof later.

4 Weak Ramanujan graphs

We recall that a graph is said to be a weak Ramanujan graph as in (14) if

\[
\lambda_1 \geq 1 - \sigma \geq \frac{1}{2}
\]

where

\[
\sigma = 2 \frac{\sum_v d_v \sqrt{d_v} - 1}{\sum_v d_v^2}.
\]  \hspace{1cm} (15)

To prove the Alon-Boppana bound, it is enough to consider only weak Ramanujan graphs.

**Lemma 5** As defined in (15), \(\sigma\) satisfies

\[
\frac{2\sqrt{d} - 1}{d} \leq \sigma \leq \frac{2\sqrt{\bar{d}} - 1}{d}
\]

where \(\bar{d}\) denotes the average degree in \(G\) and \(\bar{d}\) denote the second order degree, i.e.,

\[
\bar{d} = \frac{\sum_v d_v}{n} \quad \text{and} \quad \bar{d} = \frac{\sum_v d_v^2}{\sum_v d_v}.
\]

**Proof:** The proof is mainly by using the Cauchy-Schwarz inequality. For the upper bound, we note that

\[
\sigma = 2 \frac{\sum_v d_v \sqrt{d_v} - 1}{\sum_v d_v^2} \leq 2 \frac{\sqrt{\sum_v d_v^2} \sum_v (d_v - 1)}{\sum_v d_v^2}
\]

\[
= 2 \sqrt{\frac{\sum_v (d_v - 1)}{\sum_v d_v^2}}
\]

\[
\leq 2 \sqrt{\frac{\sum_v (d_v - 1)}{\sum_v d_v/\sqrt{n}}}
\]

\[
\leq 2 \sqrt{\frac{\sum_v (d_v - 1)}{d \sqrt{n}}} \leq 2 \sqrt{\frac{d - 1}{d}}.
\]
For the upper bound, we will use the fact that for \(a, b > 1\) and \(a + b = c\),
\[
a\sqrt{a - 1} + b\sqrt{b - 1} \geq c\sqrt{\frac{c}{2} - 1}
\]
and therefore
\[
\sum_v d_v\sqrt{d_v - 1} \geq \sum_v d_v\sqrt{\frac{\sum_v d_v}{n} - 1}.
\]
Consequently, we have
\[
\sigma = 2\frac{\sum_v d_v\sqrt{d_v - 1}}{\sum_v d_v^2} \geq 2\frac{\sum_v d_v\sqrt{\frac{\sum_v d_v}{n} - 1}}{\sum_v d_v^2 / \sum_v d_v} \geq 2\sqrt{\frac{d - 1}{d}}
\]
as desired. \(\Box\)

We remark that for graphs with average degree at least 20, we have \(\sigma < 1/2 < \lambda_1\).

**Theorem 1** Suppose a weak Ramanujan graph \(G\) has diameter \(k\). Then for any \(\epsilon > 0\), we have
\[
k \leq (1 + \epsilon)\frac{2\log \text{vol}(G)}{\log \sigma - 1}
\]
provided that the volume of \(G\) is large, i.e., \(\text{vol}(G) \geq c\sigma^{\log(\sigma)/\epsilon}\) for some small constant \(c\).

**Proof:** We set
\[
t = \left\lceil (1 + \epsilon)\frac{\log(\text{vol}(G))}{\log \sigma - 1} \right\rceil.
\]
It suffices to show that for every vertex \(v\), the ball \(B_t(v)\) has volume more than \(\text{vol}(G)/2\).

Suppose \(\text{vol}(B_t(v)) \leq \text{vol}(G)/2\). Let
\[
s_j = \frac{\text{vol}(B_j(u))}{\text{vol}(G)}.
\]
By part (i) of Lemma 4, we have \(\text{vol}(\delta(B_u(j))) \geq 0.5\text{vol}(B_u(j))\) for \(j \leq t - 1\) and therefore \(s_{j+1} \geq 1.5s_j\). Thus, if \(j \leq t - c_1 \log(\sigma^{-1})\), then \(s_j \leq \sigma^4\) where \(c_1\) is some small constant satisfying \(c_1 \leq 4(\log 1.5)^{-1}\).
Now we apply part (ii) of Lemma 4 and we have, for \( j \leq t - c_1 \log(\sigma^{-1}) \),

\[
\frac{s_{j+1}}{s_j} = \frac{\text{vol}(B_{j+1}(u))}{\text{vol}(B_j(u))} \geq \frac{\text{vol}(\delta(B_j(u)))}{\text{vol}(B_j(u))} \geq \frac{1}{(\sigma + 2s_j)^2} \geq \frac{1}{(\sigma + 2\sigma^4)}.
\]

This implies, for \( l \leq t - c_1 \log(\sigma^{-1}) \),

\[
\frac{s_l}{s_0} \geq \prod_{0 < j < l} \frac{1}{(\sigma + 2s_j)^2} \geq \prod_{0 < j < l} \frac{1}{(\sigma + 2\sigma^4)^2} \geq \frac{1}{\sigma^{2l}(1 + 2\sigma^4)^{2l}}.
\]

Since \( s_0 \geq 1/\text{vol}(G) \) and \( s_l \leq s_t \leq 1/2 \), we have

\[
\text{vol}(G) \geq \frac{1}{\sigma^{2l}(1 + 2\sigma^4)^{2l}}.
\]

Hence

\[
l \leq \frac{\log(\text{vol}(G))}{\log(\sigma^{-1}) + 2\sigma^4}.
\]

However,

\[
(1 + \epsilon) \frac{\log(\text{vol}(G))}{\log(\sigma^{-1})} \leq t \leq c_1 \log(\sigma^{-1}) + \frac{\log(\text{vol}(G))}{\log(\sigma^{-1}) + 2\sigma^4}
\]

which is a contradiction for \( G \) with \( \text{vol}(G) \) large, say, \( \text{vol}(G) \geq \sigma^{2c_1 \log \sigma / \epsilon} \). Thus we conclude that \( s_t \geq 1/2 \) and Theorem 1 is proved. \( \square \)

**Theorem 2** For a weak Ramanujan graph with diameter \( k \), for any vertex \( v \) and any \( l \leq k/4 \), the ball \( B_v(l) \) has volume at most \( \epsilon \text{vol}(G) \) if \( k \geq c \log \epsilon^{-1} \), for some constants \( c \).

**Proof:** We will prove by contradiction. Suppose that for \( j_0 = \lceil k/4 \rceil \), there is a vertex \( u \) with \( \text{vol}(B_u(j_0)) > \epsilon \text{vol}(G) \). Let \( r \) denote the largest integer such that

\[
s_r = \frac{\text{vol}(B_u(r))}{\text{vol}(G)} > \frac{1}{2}.
\]

By the assumption, we have \( r > k/4 \) and \( s_{j_0} > \epsilon \). There are two possibilities:
Case 1: $r \geq k/2$.
By part (i) of Lemma 4, we have $\text{vol}(\delta(B_u(j))) \geq 0.5\text{vol}(B_u(j))$ for $j \leq k/2$ and therefore $s_{j+1} \geq 1.5s_j$. Thus, for $j \leq k/2 - c_1 \log \epsilon^{-1}$, we have $s_j \leq \epsilon$ where $c_1 = 1/\log 1.5$. Since $k/4 \leq k/2 - c_1 \log \epsilon^{-1}$, we have a contradiction.

Case 2: $r < k/2$.
We define
\[
\bar{s}_j = \frac{\text{vol}(V \setminus B_u(j))}{\text{vol}(G)}.
\]
Thus $\bar{s}_j < 1/2$ for all $j \geq k/2$. We consider two subcases.

Subcase 2a: Suppose $\bar{s}_j \geq \epsilon$ for $j \geq k/2$.
Using Lemma 4, for $j$ where $r \leq j \leq k/2$, we have $\bar{s}_j \geq 1.5\bar{s}_{j+1}$. Thus, for some $j_1 \geq k/2 - c_1 \log \epsilon^{-1}$, we have $s_j \geq 1/2$ or equivalently, $s_j \leq \epsilon$. By using Lemma 4 again, for $j \leq j_1$, we have $s_{j+1} \geq 1.5s_j$ and therefore for any $j \leq j_1 - c_1 \log \epsilon^{-1}$ we have $s_j \leq \epsilon$. Since $j_1 - c_1 \log \epsilon^{-1} \geq k/2 - 2c_1 \log \epsilon^{-1} \geq k/4$, we again have a contradiction to the assumption $s_{j_0} \geq \epsilon$.

Subcase 2b: Suppose $\bar{s}_j < \epsilon$ for $j \geq k/2$.
We apply part (ii) of Lemma 4 and we have, for $j \geq k/2$,
\[
\frac{\bar{s}_j}{\bar{s}_{j+1}} \geq \frac{1}{(\sigma + 2\epsilon)^2}.
\]
This implies, for $j_2 = \lceil k/2 \rceil$,
\[
\frac{\bar{s}_{j_2}}{\bar{s}_k} \geq \prod_{k/2 < j \leq k} \frac{1}{(\sigma + 2s_j)^2} \geq \frac{1}{(\sigma + 2\epsilon)^k}.
\]
Since $\bar{s}_k \geq 1/\text{vol}(G)$, we have
\[
\bar{s}_{j_1} \geq \frac{1}{\text{vol}(G)(\sigma + 2\epsilon)^{j_1}}.
\]
Since the assumption of this subcase is $\bar{s}_{j_1} < \epsilon$, we have
\[
k \geq \frac{\log n + \log \epsilon^{-1}}{\log \sigma^{-1}}.
\]
We now use Lemma 4 and we have, for $j = k/2 - j' \geq r$
\[
\bar{s}_j \geq \frac{1}{\text{vol}(G)(\sigma + 2\epsilon)^{k+2j'}}.
\]
Therefore, for some $j \leq k/2 - \log \epsilon^{-1}/\log \sigma^{-1}$, we have $\bar{s}_j > 1/2$ which implies $r \geq k/2 - \log \epsilon^{-1}/\log \sigma^{-1}$.

Now we use the same argument as in Case 1 except shifting $r$ by $\log \epsilon^{-1}/\log \sigma^{-1}$. For some $j \leq r - c_1 \log \epsilon^{-1} \leq k/2 - \log \epsilon^{-1}/\log \sigma^{-1} - c_1 \log \epsilon^{-1}$, we have $s_j < \epsilon$. Since $\log \epsilon^{-1}/\log \sigma^{-1} + c_1 \log \epsilon^{-1} < k/4$, this leads to a contradiction and Theorem 2 is proved.

\[ \square \]

5 Non-backtracking random walks

Before we proceed to the proof of the Alon-Boppana bound, we will need some basic facts on non-backtracking random walks.

A non-backtracking walk is a sequence of vertices $p = (v_0, v_1, \ldots, v_t)$ for some $t$ such that $v_{i-1} \sim v_i$ and $v_{i+1} \neq v_{i-1}$ for $i = 1, \ldots, t - 2$. The non-backtracking random walk can be described as follows: For $i \geq 1$, at the $i$th step on $v_i$, choose with equal probability a neighbor $u$ of $v_i$ where $u \neq v_{i-1}$, move to $u$ and set $v_{i+1} = u$. To simplify notation, we call a non-backtracking walk an NB-walk. The modified transition probability matrix $\tilde{P}_k$, for $k = 0, 1, \ldots, t - 1$, is defined by

\[
\tilde{P}_k(u, v) = \begin{cases} 
P^k(u, v) & \text{if } k = 0 \\
\sum_{p \in \mathcal{P}^{(k)}_{u,v}} w(p) & \text{if } k \geq 1 \end{cases}
\]

(16)

where the weight $w(p)$ for an NB-walk $p = (v_0, v_1, \ldots, v_t)$ with $t \geq 1$ is defined to be

\[
w(p) = \frac{1}{d_{v_0} \prod_{i=1}^{t-1} (d_{v_i} - 1)}
\]

(17)

and $\mathcal{P}^{(k)}_{u,v}$ denotes the set of non-backtracking walks from $u$ to $v$. For a walk $p = (v_0)$ of length 0, we define $w(p) = 1$.

Although a non-backtracking random walk is not a Markov chain, it is closely related to an associated Markov chain as we will describe below (also see [6]).

For each edge $\{u, v\}$ in $E$, we consider two directed edges $(u, v)$ and $(v, u)$. Let $\hat{E}$ denote the set consisting of all such directed edges, i.e. $\hat{E} = \{(u, v) : \{u, v\} \in E\}$. We consider a random walk on $\hat{E}$ with transition
probability matrix $P$ defined as follows:

$$P((u,v),(u',v')) = \begin{cases} \frac{1}{d_u} & \text{if } v = u' \text{ and } u \neq v' \\ 0 & \text{otherwise.} \end{cases}$$

Let $1_E$ denote the all 1’s function defined on the edge set $E$ as a row vector. From the above definition, we have

$$1_E P = 1_E. \quad (18)$$

In addition, we define the vertex-edge incidence matrix $B$ and $B^*$ for $a \in V$ and $(b,c) \in \hat{E}$ by

$$B(a,(b,c)) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise} \end{cases}$$

$$B^*((b,c),a)) = \begin{cases} 1 & \text{if } c = a, \\ 0 & \text{otherwise.} \end{cases}$$

Let $1_V$ denote all 1’s vector defined on the vertex set $V$. Then

$$1_V B = 1_E. \quad (19)$$

Although $\hat{P}_k$ is not a Markov chain, it is related to the Markov chain determined by $P$ on $\hat{E}$ as follows:

**Fact 1:** For $l \geq 1$,

$$\hat{P}_l = D^{-1} B P^l B^* \quad (20)$$

and for the case of $l = 0$, we have $\hat{P}_0 = I$.

By combining (19) and (20), we have

**Fact 2:**

$$1_V D \hat{P}_l = 1_E B^* = 1_V D. \quad (21)$$

Note that $1_V D$ is just the degree vector for the graph $G$. Therefore (21) states that the degree vector is an eigenvector of $\hat{P}_l$. Using Fact 1 and 2, we have the following:
Lemma 6

(i) For a fixed vertex $x$ and any integer $j \geq 0$, we have

$$\sum_u d_u \sum_{p \in \mathcal{P}(j)} w(p) = d_x$$

(ii) For a fixed vertex $u$, we have

$$\sum_x \sum_{p \in \mathcal{P}(j)} w(p) = 1_{u}(I + \tilde{P}_1 + \ldots + \tilde{P}_l) = l + 1$$

where $1_u$ denotes the characteristic function which assumes value 1 at $u$ and 0 elsewhere.

Proof: The proof of (22) and (23) follows from the fact that

$$1_V D \tilde{P}_j(x) = 1_V D (D^{-1}BP^j B^*) = 1_E P^j B^* = 1_E B^* = 1_V D(x)$$

and $1_u \tilde{P}_j(x) = w(p)$ for $p \in \mathcal{P}(j)$.

6 An Alon-Boppana bound for $\lambda_1$

Theorem 3 In a graph $G = (V,E)$ with diameter $k$, the first nontrivial eigenvalue $\lambda_1$ satisfies

$$\lambda_1 \leq 1 - \sigma \left(1 - \frac{c}{k}\right)$$

where $\sigma$ is as defined in (15), provided $k \geq c' \log \sigma^{-1}$ and $\text{vol}(G) \geq c'' \sigma \log \sigma$ for some absolute constants $c$'s.

Proof: If $G$ is not a weak Ramanujan graph, we have $\lambda_1 \leq 1 - \sigma$ and we are done. We may assume that $G$ is weak Ramanujan.

From the definition of $\lambda_1$, we have

$$\lambda_1 \leq \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x f^2(x) d_x} = R(f)$$

where $f$ satisfies $\sum_x f(x) d_x = 0$.

We will construct an appropriate $f$ satisfying $R(f) \leq 1 - \sigma(1 - c/k)$ and therefore serve as an upper bound for $\lambda_1$. We set

$$t = \left\lfloor \frac{\log(\text{vol}(G))}{\log \sigma^{-1}} \right\rfloor$$
and choose $\epsilon$ satisfying

$$\epsilon \leq \frac{\sigma}{t} \leq \frac{c\sigma}{k}$$

by using Theorem 1 where $\sigma$ is as defined in (15).

We consider a family of functions defined as follows. For a specified vertex $u$ and an integer $l = \lfloor k/4 \rfloor$, we consider a function $g_u : V \to \mathbb{R}^+$, defined by

$$g_u(x) = \left(1_u(I + \tilde{P}_1 + \ldots + \tilde{P}_l)(x)\right)^{1/2}$$

$$= \left(\sum_{j=0}^{l} \sum_{p \in \mathcal{P}_u^{(j)}} w(p)\right)^{1/2}$$

where $\tilde{P}_j$ is as defined in (20) and $1_u$ is treated as a row vector. In other words, $g_u$ denotes the square root of the sum of non-backtracking random walks starting from $u$ taking $i$ steps for $i$ ranging from 0 to $l$.

Claim A:

$$\sum_{u} d_u \sum_{x} g_u^2(x)d_x = \sum_{j=0}^{l} \sum_{x} \sum_{p \in \mathcal{P}_u^{(j)}} d_u w(p)d_x = (l + 1) \sum_{x} d_x^2$$

where the weight $w(p)$ of a walk $p$ is as defined in (17).

Proof of Claim A: From the definition of $g_u$ and (16), we have
Claim A is proved.

\[ \sum_u d_u \sum_x g_u^2(x)dx = \sum_{j=0}^l \sum x \sum_{p \in \mathcal{P}^{(l)}} d_u w(p) \]

\[ = \sum_u d_u 1_u B(I + \tilde{P}_1 + \ldots + \tilde{P}_l)(x) \]

\[ = \sum_{i=1}^l \sum u 1_u BP^i B^*(x)dx + \sum x d_x^2 \]

\[ = \sum_{i=1}^l 1_E P^i B^*(x)dx + \sum x d_x^2 \]

\[ = (l + 1) \sum x d_x^2. \]

Claim B:

\[ \sum_u d_u \sum_{x \sim y} (g_u(x) - g_u(y))^2 \leq (l + 1 - l\sigma) \sum x d_x^2. \]

where \( \sum_{x \sim y} \) denotes the sum ranging over unordered pairs \( \{x, y\} \) where \( x \) is adjacent to \( y \).

**Proof of Claim B:**

We will use the following fact for \( a_i, b_i > 0 \).

\[ \left( \sqrt{\sum_i a_i} - \sqrt{\sum_i b_i} \right)^2 \leq \sum_i \left( \sqrt{a_i} - \sqrt{b_i} \right)^2 \tag{25} \]

which can be easily checked.
For a fixed vertex \( u \), we apply Claim B:

\[
\sum_{x \sim y} (g_u(x) - g_u(y))^2
= \sum_{x \sim y} \left( \sqrt{\sum_{p \in \mathcal{P}_{u,x}^{(t)}} w(p)} - \sqrt{\sum_{p' \in \mathcal{P}_{u,y}^{(t)}} w(p')} \right)^2
\leq \sum_{t \leq l-1} \sum_{x \in V} \sum_{p \in \mathcal{P}_{u,x}^{(t)}} \left( \sqrt{w(p)} - \sqrt{w(p')} \right)^2 + \sum_{p \in \mathcal{P}_{u,x}^{(l)}} w(p)(d_x - 1)
\leq \sum_{t \leq l-1} \sum_{x \in V} \sum_{p \in \mathcal{P}_{u,x}^{(t)}} \left( \sqrt{w(p)} - \sqrt{w(p')} \right)^2 (d_x - 1) + \sum_{p \in \mathcal{P}_{u,x}^{(l)}} w(p)(d_x - 1)
\leq \sum_{t \leq l-1} \sum_{x \in V} \sum_{p \in \mathcal{P}_{u,x}^{(t)}} \left( \sqrt{w(p)} - \sqrt{w(p')} \right)^2 (d_x - 1) + \sum_{p \in \mathcal{P}_{u,x}^{(l)}} w(p)(d_x - 1)
\leq \sum_{t \leq l-1} \sum_{x \in V} \sum_{p \in \mathcal{P}_{u,x}^{(t)}} \left( \sqrt{w(p)} - \sqrt{w(p')} \right)^2 (d_x - 1) + \sum_{p \in \mathcal{P}_{u,x}^{(l)}} w(p)(d_x - 1)
\]

Using Fact 3, we have

\[
\sum_u d_u \sum_{x \sim y} (g_u(x) - g_u(y))^2
\leq \sum_{t \leq l-1} \sum_u \sum_{p \in \mathcal{P}_{u,x}^{(t)}} w(p)(d_x - 2\sqrt{d_x} - 1) + \sum_u \sum_{p \in \mathcal{P}_{u,x}^{(l)}} w(p)(d_x - 1)
\leq \sum_{t \leq l-1} \sum_{x \in V} \sum_{p \in \mathcal{P}_{u,x}^{(t)}} w(p)(d_x - 2\sqrt{d_x} - 1) + \sum_{p \in \mathcal{P}_{u,x}^{(l)}} w(p)(d_x - 1)
\leq \sum_{t \leq l-1} \sum_{x \in V} \sum_{p \in \mathcal{P}_{u,x}^{(t)}} w(p)(d_x - 2\sqrt{d_x} - 1) + \sum_{p \in \mathcal{P}_{u,x}^{(l)}} w(p)(d_x - 1)
\]

This proves Claim B.

Claim C: There is a vertex \( u \) satisfying

\[
R(g_u) \leq 1 - \sigma \left( 1 - \frac{1}{l+1} \right)
\]
Proof of Claim C:

Combining Claim A and B, we have

\[
\sum_u d_u \sum_{x \sim y} (g_u(x) - g_u(y))^2 \\
\leq (l + 1 - l\sigma) \sum_x d_x^2 \\
\leq (l + 1 - l\sigma) \left( \frac{1}{l+1} \right) \sum_u d_u \sum_x g_u^2(x) d_x \\
= \left( 1 - \frac{l\sigma}{l+1} \right) \sum_u d_u \sum_x g_u^2(x) d_x
\]

(26)

Thus we deduce that there is a vertex \( u \) such that

\[
R(g_u) = \frac{\sum_{x \sim y} (g_u(x) - g_u(y))^2}{\sum_x g_u^2(x) d_x} \\
\leq 1 - \frac{l\sigma}{l+1}.
\]

(27)

We define

\[
\alpha_v = \frac{\sum_x g_v(x) d_x}{\sum_x d_x} = \frac{\sum_x g_v(x) d_x}{\text{vol}(G)}
\]

We consider the function \( g'_u \) defined by

\[
g'_u(x) = g_u(x) - \alpha_u
\]

Clearly, \( g'_u \) satisfies the condition that

\[
\sum_x g'_u(x) d_x = 0
\]

Hence, we have

\[
\lambda_1 \leq R(g'_u) = \frac{\sum_{x \sim y} (g'_u(x) - g'_u(y))^2}{\sum_x g'_u^2(x) d_x} \\
= \frac{\sum_{x \sim y} (g_u(x) - g_u(y))^2}{\sum_x g_u^2(x) d_x - \alpha_u^2 \text{vol}(G)}.
\]

(28)

Note that by the Cauchy-Schwarz inequality, we have

\[
\left( \sum_{x \in B_u(l)} g_u(x) d_x \right)^2 \leq \text{vol}(B_u(l)) \sum_{x \in B_u(l)} g_u^2(x) d_x.
\]
and therefore

\[ \alpha_u^2 \leq \frac{\text{vol}(B_u(l))}{\text{vol}(G)^2} \sum_x g_u^2(x) dx. \]

By substitution into (28) and using (35), we have

\[ \lambda_1 \leq R(g'_u) \leq \frac{R(g)}{1 - \frac{\text{vol}(B_u(l))}{\text{vol}(G)}} \leq 1 - \sigma \left(1 - \frac{1}{t+1}\right) \]

\[ \leq 1 - \sigma \left(1 - \frac{1}{t+1}\right) + \frac{\text{vol}(B_u(l))}{\text{vol}(G)} \]

\[ \leq 1 - \sigma \left(1 - \frac{c}{t+1}\right) \]

The last inequality follows from Theorem 2 and the choice of \( \epsilon = \sigma/k \). This completes the proof of Theorem 3. □

7 A lower bound for \( \lambda_{n-1} \)

If a graph is bipartite, it is known (see [2]) that \( \lambda_i = 2 - \lambda_{n-i-1} \) for all \( 0 \leq i \leq n-1 \) and, in particular, \( \lambda_{n-1} = 2 - \lambda_0 = 2 \). If \( G \) is not bipartite, it is easy to derive the following lower bound:

\[ \lambda_{n-1} \geq 1 + 1/(n-1) \]

by using the fact that the trace of \( L \) is \( n \). This lower bound is sharp for the complete graph. However if \( G \) is not the complete graph, is it possible to derive a better lower bound? The answer is affirmative. Here we give an improved lower bound for \( \lambda_{n-1} \).

**Theorem 4** In a connected graph \( G = (V, E) \) with diameter \( k \), the largest eigenvalue \( \lambda_{n-1} \) of the normalized Laplacian \( L \) of \( G \) satisfies

\[ \lambda_{n-1} \geq 1 + \sigma \left(1 - \frac{c}{k}\right) \]

where \( \sigma \) is as defined in (15), provided \( k \geq c' \log \sigma^{-1} \) and \( \text{vol}(G) \geq c'' \sigma \log \sigma \) for some absolute constants \( c \)'s.

**Proof:** By definition, \( \lambda_{n-1} \) satisfies

\[ \lambda_{n-1} \geq \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x f^2(x) dx} = R(f) \]

(33)
for any $f : V \rightarrow \mathbb{R}$.

We will construct an appropriate $f$ such that $R(f) \geq 1 + \sigma(1 - c/\gamma)$ by considering the following function $f_u : V \rightarrow \mathbb{R}^+$, for a fixed vertex $u$, defined by

$$
\eta_u(x) = \begin{cases} 
(-1)^t \chi_u(\tilde{P}_t(x))^{-1/2} & \text{if dist}(u, x) = t \leq l \\
0 & \text{otherwise}
\end{cases}
$$

where $l \leq \gamma/2$. Note that $|\eta_u(x)| = g_u(x)$ since we assume that $l \leq \gamma/2$.

Using the same proof in Claim A, we have

Claim A':

$$
\sum_u d_u \sum_x \eta_u^2(x) d_x = \sum_{t=0}^l \sum_x \sum_{p \in \mathcal{P}_u^{(t)}} d_u w(p) d_x = (l + 1) \sum_x d_x^2.
$$

Claim B':

$$
\sum_u d_u \sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2 \geq (l + 1 + l\sigma) \sum_x d_x^2.
$$

Proof of Claim B':

The proof is quite similar to that of Claim B. For a fixed vertex $u$, the sum over unordered pair $\{x, y\}$ where $x \sim y$,

$$
\sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2
\leq \sum_{t \leq l-1} \sum_{x \in V} \sum_{p \in \mathcal{P}_u^{(t)}} \left( \sqrt{w(p)} + \sqrt{w(p')} \right)^2 - \sum_{p \in \mathcal{P}_u^{(t)}} w(p)(d_x - 1)
\leq \sum_{t \leq l-1} \sum_{x \in V} \sum_{p \in \mathcal{P}_u^{(t)}} \left( \sqrt{w(p)} + \sqrt{w(p')} \right)^2 (d_x - 1) - \sum_{p \in \mathcal{P}_u^{(t)}} \sqrt{w(p)}(d_x - 1)
\leq \sum_{t \leq l-1} \sum_{x \in V} \sum_{p \in \mathcal{P}_u^{(t)}} w(p) \left( 1 + \frac{1}{d_x - 1} + \frac{2}{\sqrt{d_x - 1}} \right)(d_x - 1) - \sum_{p \in \mathcal{P}_u^{(t)}} w(p)(d_x - 1)
\leq \sum_{t \leq l-1} \sum_{x \in V} \sum_{p \in \mathcal{P}_u^{(t)}} w(p) \left( d_x + 2\sqrt{d_x - 1} \right) - \sum_{p \in \mathcal{P}_u^{(t)}} w(p)(d_x - 1).
$$
Using Fact 3, we have
\[
\sum_u d_u \sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2 \\
\geq \sum_{t \leq l-1} \sum_u d_u \sum_{p \in \mathcal{P}_{u,x}^{(t)}} w(p)(d_x + 2\sqrt{d_x - 1}) - \sum_u d_u \sum_{p \in \mathcal{P}_{u,x}^{(l)}} w(p)(d_x - 1) \\
= l \sum_x d_x (d_x + 2\sqrt{d_x - 1}) - \sum_x d_x^2 \\
= l(1 + \sigma) \sum_x d_x^2 - \sum_x d_x^2 \\
= (l - 1 + l\sigma) \sum_x d_x^2
\]
This proves Claim B'.

Combining Claims A' and B', we have
\[
\sum_u d_u \sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2 \\
\geq (l - 1 + l\sigma) \sum_x d_x^2 \\
\geq (l - 1 + l\sigma) \left( \frac{1}{l+1} \right) \sum_u d_u \sum_x \eta_u^2(x) d_x \\
= \left( 1 + \frac{l\sigma}{l-1} \right) \sum_u d_u \sum_x \eta_u^2(x) d_x
\]
(34)
Thus we deduce that there is a vertex \( u \) such that
\[
R(\eta_u) = \frac{\sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2}{\sum_x \eta_u^2(x) d_x} \\
\leq 1 + \frac{l\sigma}{l-1}.
\]
(35)
We consider the function \( \eta'_u \) defined by
\[
\eta'_u(x) = \eta_u(x) - \alpha_u
\]
where
\[
\alpha_u = \frac{\sum_x \eta_u(x) d_x}{\sum_x d_x} = \frac{\sum_x \eta_u(x) d_x}{\text{vol}(G)}
\]
so that $\eta_u'$ satisfies the condition that

$$\sum_x \eta_u'(x)d_x = 0$$

Hence, we have

$$\lambda_{n-1} \geq R(\eta_u') = \frac{\sum_{x \sim y} (\eta_u'(x) - \eta_u'(y))^2}{\sum_x \eta_u'^2(x)d_x}$$

$$= \frac{\sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2}{\sum_x \eta_u^2(x)d_x - \alpha_u^2 \text{vol}(G)}$$

$$\geq 1 + \sigma (1 + \frac{c}{l}) - \frac{\text{vol}(B_u(l))}{\text{vol}(G)}.$$

This completes the proof of Theorem 4. $\square$

References


