A spectral Turán theorem

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Abstract

If all nonzero eigenvalues of the (normalized) Laplacian of a graph G are close to 1, then G is t-Turán in the sense that any subgraph of G containing no K_{t+1} contains at most $(1 - 1/t + o(1))e(G)$ edges where $e(G)$ denotes the number of edges in G.

1 Introduction

One of the classical theorems in graph theory is Turán's Theorem which states that a graph on n vertices containing no K_{t+1} can have at most $(1 - 1/t + o(1))\binom{n}{2}$ edges. Sudakov, Szabó and Vu [6] consider a generalization of Turán's Theorem. A graph G is said to be t-Turán if any subgraph of G containing no K_{t+1} has at most $(1-1/t+o(1))e(G)$ edges where $e(G)$ denotes the number of edges in G. In [6], it is shown that a regular graph on n vertices with degree d is t-Turán if the second largest eigenvalue of its adjacency matrix λ is sufficiently small.

In this paper, we consider Turán numbers for general graphs as introduced in [6]. For two given graphs G and H, the Turán number $t(G, H)$ is defined to be

$$t(G, H) = \max\{e(G')| \ : \ G' \ \text{ is a subgraph of } G \text{ containing no } H\}.$$

The classical Turán number is the special case that G is a complete graph K_n. Turán's theorem implies

$$t(K_n, K_{t+1}) = (\frac{t-1}{t} + o(1))\binom{n}{2}.$$

In this paper, we will show that

$$t(G, K_{t+1}) = (\frac{t-1}{t} + o(1))e(G) \qquad (1)$$

as long as certain spectral bounds of G are satisfied (to be specified in Section 4).

Since any t-partite subgraph of G contains no K_{t+1}, the inequality $t(G, K_{t+1}) \geq (1-\frac{1}{t}+o(1))e(G)$ always holds. Thus, equation (1) implies that a maximum t-partite subgraph of G is an extremal graph having the

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maximum number of edges among all subgraphs of G containing no K_{t+1}. In section 4, we will show that our main theorem implies (the asymptotical version of) the classical Turán's Theorem as a special case. Another consequence of our main theorem is the result in [6] for d-regular graphs. Namely, if the second largest eigenvalue μ of the adjacency matrix of a d-regular graph on n vertices satisfies $\mu \ll d^t/n^{t-1}$, then $t(G, K_{t+1}) = (1 - 1/t + o(1))dn/2$. This will also be proved in Section 4.

In order to derive the relationship between the spectral bounds and the Turán property, we will first consider eigenvalues of the (normalized) Laplacian. Detailed definitions will be given in the next section.

The connection between eigenvalues of the Laplacian and Turán numbers depends on a notion of generalized volumes: For a subset X of vertices in a graph G, the k-volume of X is defined by

$$\mathrm{vol}_k(X) = \sum_{v \in X} d_v^k,$$

where d_v denotes the degree of v in G. We will first describe several key properties of graphs which are consequences of spectral gaps. In particular, we will give several general isoperimetric inequalities in Section 3. These inequalities provide good estimates for the "discrepancies" of a graph. We will use these inequalities to establish the relationship between eigenvalues and the Turán property. We will show that if the non-zero eigenvalues of the (normalized) Laplacian are bounded (depending on t and the volumes of G), then the graph is t-Turán. The proofs are given in Section 4.

2 Preliminaries on eigenvalues

For a graph G, there are several ways to evaluate eigenvalues by associating various matrices with G. A typical matrix is the adjacency matrix $A = A_G$ which has entries $A(u, v) = 1$ if u and v are adjacent, and 0 otherwise. Another matrix is the combinatorial Laplacian L which is defined as $L = D - A$ where D is the diagonal matrix with diagonal entries $D(v, v) = d_v$ where d_v is the degree of the vertex v. The well-known matrix-tree theorem of Kirchhoff [4] states that the number of spanning trees in a graph G is the product of all (except for the smallest) eigenvalues of L divided by the number of vertices of G. The eigenvalues of the adjacency matrix are useful in enumerating walks in a graph. For example, the largest eigenvalue of A, denoted by $\|A\|$, is the limit of the k-th root of the number of k-walks in G, as k approaches infinity. In this paper, we will mainly focus on the (normalized) Laplacian $\mathcal{L}$, which is defined as follows:

$$\mathcal{L}(u, v) = \begin{cases} 1 - \frac{A(v,v)}{d_v} & \text{if } u = v \text{ and } d_v \neq 0 \\ -\dfrac{1}{\sqrt{d_u d_v}} & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

We can write

$$\mathcal{L} = D^{-1/2} L D^{-1/2}$$

with the convention $D^{-1}(v,v) = 0$ for $d_v = 0$.

For a regular graph with degree d, we have

$$\mathcal{L} = I - \frac{1}{d} A.$$

Let g denote an arbitrary function which assigns to each vertex v of G a real value $g(v)$. We can view g as a column vector. Then

$$\begin{aligned}
\frac{\langle g, \mathcal{L} g\rangle}{\langle g, g\rangle} &= \frac{\langle g, D^{-1/2} L D^{-1/2} g\rangle}{\langle g, g\rangle} \\
&= \frac{\langle f, Lf\rangle}{\langle D^{1/2} f, D^{1/2} f\rangle} \\
&= \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v} \qquad (2)
\end{aligned}$$

where $g = D^{1/2} f$ and $\sum_{u \sim v}$ denotes the sum over all unordered pairs $\{u, v\}$ for which u and v are adjacent. Here $\langle f, g\rangle = \sum_x f(x) g(x)$ denotes the standard inner product in $\mathbb{R}^n$. (We note that we can also use the inner product $\langle f, g\rangle = \sum \overline{f(x)} g(x)$ for complex-valued functions.) From equation (2), we see that all eigenvalues are non-negative and 0 is an eigenvalue of $\mathcal{L}$. We denote the eigenvalues of $\mathcal{L}$ by $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$. Let $\mathbf{1}$ denote the constant function which assumes the value 1 on each vertex. Then $D^{1/2}\mathbf{1}$ is an eigenfunction of $\mathcal{L}$ with eigenvalue 0.

Quite a few basic facts can be derived from the above definition (see [2]). All λ_i are between 0 and 2. The number of eigenvalues of $\mathcal{L}$ having value 0 is the same as the number of connected components in G. The maximum eigenvalue of $\mathcal{L}$ is 2 if and only if the graph is bipartite. In the next few sections, we will focus on the family $\mathcal{F}_\delta$ of graphs with Laplacian eigenvalues satisfying

$$\bar{\lambda} = \max_{i \neq 0} |1 - \lambda_i| < \delta \qquad (3)$$

for $i \neq 0$. We note that for d-regular graphs, the eigenvalues of the adjacency matrix are just $d(1 - \lambda_i)$ so the so-called (n, d, λ)-graphs in [1, 5] are in $\mathcal{F}_\delta$ for $\delta = \bar{\lambda}/d$.

3 Eigenvalues and discrepancies

A main tool for investigating various graph invariants for $\mathcal{F}_\delta$ concerns the notion of *discrepancy* and the related discrepancy inequalities. A typical definition for discrepancy is the difference between the *actual* *quantity* and the *expected value*. The goal is to upper bound the discrepancy in terms of eigenvalues. For example, in a given graph G, a quantity of concern is the number $e(X,Y)$ of edges between two subsets X and Y. In many situations (such as G is regular), the expected value of $e(X,Y)$ is taken to be the edge density multiplied by the cardinality of X and Y. The condition of the graph being regular is quite restrictive. In particular, such an inequality cannot be applied to (non-regular) subgraphs of a regular graph. Here we extend such a discrepancy inequality to general graphs by using the eigenvalues of the Laplacian.

Lemma 1 *Suppose a graph G on n vertices has eigenvalues λ_i of the Laplacian satisfying $\bar{\lambda} = \max_{i\neq 0} |1-\lambda_i| < \delta$. For any two subsets X and Y of vertices, $e(X,Y)$ denotes the number of ordered pairs (x,y) so that $\{x,y\}$ is an edge and $x \in X$ and $y \in Y$. Then $e(X,Y)$ satisfies*

$$|e(X,Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)}| \leq \delta\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)}$$

where $\mathrm{vol}(X) = \sum_{x\in X} d_x$ *and* $\mathrm{vol}(G) = \sum_v d_v$.

The above lemma is a special case of the following:

Lemma 2 *Suppose k is a given real value (possibly negative) and a graph G on n vertices has Laplacian eigenvalues λ_i satisfying $\bar{\lambda} = \max_{i\neq 0} |1-\lambda_i| < \delta$. Then for any two subsets X and Y of vertices, the k-weight of X and Y, denoted by*

$$e_k(X,Y) = \sum_{u\in X} \sum_{v \in Y, v\sim u} d_u^k d_v^k$$

satisfies

$$|e_k(X,Y) - \frac{\mathrm{vol}_{k+1}(X)\mathrm{vol}_{k+1}(Y)}{\mathrm{vol}(G)}| \leq \delta\sqrt{\mathrm{vol}_{2k+1}(X)\mathrm{vol}_{2k+1}(Y)}$$

where $\mathrm{vol}_i(X) = \sum_{x\in X} d_x^i$ *and* $\mathrm{vol}_i(G) = \sum_v d_v^i$.

Proof: We define

$$\psi_X(u) = \begin{cases} d_u^k & \text{if } u \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} |e_k(X,Y) &- \frac{\mathrm{vol}_{k+1}(X)\mathrm{vol}_{k+1}(Y)}{\mathrm{vol}(G)}| \\ &= |\langle \psi_X, A\psi_Y\rangle - \frac{\mathrm{vol}_{k+1}(X)\mathrm{vol}_{k+1}(Y)}{\mathrm{vol}(G)}| \\ &= |\langle \psi_X, D^{1/2}(I-\mathcal{L})D^{1/2}\psi_Y\rangle - \frac{\mathrm{vol}_{k+1}(X)\mathrm{vol}_{k+1}(Y)}{\mathrm{vol}(G)}| \\ &= |\langle \psi_X, D^{1/2}(I-\mathcal{L}-\phi_0^*\phi_0)D^{1/2}\psi_Y\rangle| \end{aligned}$$

since the eigenvector ϕ_0 associated with eigenvalue 0 has coordinates $\sqrt{d_v/\mathrm{vol}(G)}$, where f^* denotes the transpose of f. Since G is in $\mathcal{F}_\delta$, we have

$$\|I - \mathcal{L} - \phi_0^*\phi_0\| \leq \delta.$$

Therefore

$$\begin{aligned} |e_k(X,Y) - \frac{\mathrm{vol}_{k+1}(X)\mathrm{vol}_{k+1}(Y)}{\mathrm{vol}(G)}| &\leq \|D^{1/2}\psi_X\| \;\; \|I - \mathcal{L} - \phi_0^*\phi_0\| \;\; \|D^{1/2}\psi_Y\| \\ &= \sqrt{\mathrm{vol}_{2k+1}(X)}\; \delta \; \sqrt{\mathrm{vol}_{2k+1}(Y)} \end{aligned}$$

as desired. □

For a vertex v in a graph G, the neighborhood $\Gamma(v)$ of v is defined by

$$\Gamma(v) = \{u \; : \; u \sim v\} = \{u \; : \; \{u, v\} \text{ is an edge.}\}$$

In general, the neighborhood $\Gamma_X(v)$ of v in X is denoted by

$$\Gamma_X(v) = \{u \in X \; : \; u \sim v\}.$$

In addition to Lemma 2, we also need the following estimate:

Lemma 3 *Suppose a graph G on n vertices has Laplacian eigenvalues λ_i satisfying $\bar{\lambda} = \max_{i \neq 0} |1 - \lambda_i| < \delta$. For any real value k and any subset X of vertices of G, we have*

$$\sum_{v \in X} \frac{1}{d_v}\left(\mathrm{vol}_k(\Gamma_X(v)) - d_v \frac{\mathrm{vol}_{k+1}(X)}{\mathrm{vol}(G)}\right)^2 \leq \delta^2 \mathrm{vol}_{2k+1}(X)$$

where $\Gamma_X(v) = \{u \in X : u \sim v\}$.

Proof: We consider ψ_X as defined in the proof of Lemma 2. The difference of $(I - \mathcal{L})D^{1/2}\psi_X$ and the projection of $D^{1/2}\psi_X$ on $D^{1/2}\mathbf{1}$ can be written as:

$$\|D^{-1/2}A\psi_X - \langle D^{1/2}\psi_X, D^{1/2}\mathbf{1}\rangle \frac{D^{1/2}\mathbf{1}}{\mathrm{vol}(G)}\| \leq \delta \|D^{1/2}\psi_X\|$$

which implies

$$\sum_{v \in X} \frac{1}{d_v}\left(\mathrm{vol}_k(\Gamma_X(v)) - d_v \frac{\mathrm{vol}_{k+1}(X)}{\mathrm{vol}(G)}\right)^2 \leq \delta^2 \mathrm{vol}_{2k+1}(X)$$

as desired. □

Lemma 4 *Suppose a graph G on n vertices has Laplacian eigenvalues λ_i satisfying $\bar{\lambda} = \max_{i \neq 0} |1 - \lambda_i| < \delta$. Suppose X is a subset of vertices of G and v is a vertex in X. Let $\Gamma_X(v)$ denote the neighborhood of v in X and let $R(v)$ denote a subset of $\Gamma(v)$. We have*

$$\sum_{v \in X} \frac{|\Gamma_X(v)|^2}{d_v} \geq \frac{\mathrm{vol}^3(X)}{\mathrm{vol}^2(G)} + O(\delta \frac{\mathrm{vol}^2(X)}{\mathrm{vol}(G)}) + O(\delta^2 \mathrm{vol}(X)). \tag{4}$$

and

$$\sum_{v \in X} \frac{|\Gamma_X(v)| \ \ |R(v)|}{d_v} \leq \sum_{v \in X} \frac{|R(v)| \mathrm{vol}(X)}{\mathrm{vol}(G)} + O(\delta \frac{\mathrm{vol}^2(X)}{\mathrm{vol}(G)}) + O(\delta^2 \mathrm{vol}(X)) \tag{5}$$

Proof: Using Lemma 3, we have

$$\begin{aligned} \sum_{v \in X} \frac{|\Gamma_X(v)|^2}{d_v} &= \sum_{v \in X} \frac{d_v \mathrm{vol}^2(X)}{\mathrm{vol}^2(G)} + \sum_{v \in X} \frac{|\Gamma_X(v)|^2 - d_v^2 \mathrm{vol}^2(X)/\mathrm{vol}^2(G)}{d_v} \\ &\geq \frac{\mathrm{vol}^3(X)}{\mathrm{vol}^2(G)} + O(\delta \frac{\mathrm{vol}^2(X)}{\mathrm{vol}(G)}) + O(\delta^2 \mathrm{vol}(X)). \end{aligned}$$

$$\begin{aligned} \sum_{v \in X} \frac{|\Gamma_X(v)| \ \ |R(v)|}{d_v} &= \sum_{v \in X} \frac{|R(v)| \mathrm{vol}(X)}{\mathrm{vol}(G)} + \sum_{v \in X} \frac{|R(v)|(|\Gamma_X(v)| - d_v \mathrm{vol}(X)/\mathrm{vol}(G))}{d_v} \\ &\leq \sum_{v \in X} \frac{|R(v)| \mathrm{vol}(X)}{\mathrm{vol}(G)} + \delta \sqrt{\mathrm{vol}(X) (\sum_{v \in X} \frac{|R(v)|^2}{d_v})} \\ &\leq \sum_{v \in X} \frac{|R(v)| \mathrm{vol}(X)}{\mathrm{vol}(G)} + O(\delta \frac{\mathrm{vol}(X)^2}{\mathrm{vol}(G)}) + O(\delta^2 \mathrm{vol}(X)). \end{aligned}$$

□

A useful generalization of Lemma 4 is the following:

Lemma 5 *Suppose a graph G on n vertices has Laplacian eigenvalues λ_i satisfying $\bar{\lambda} = \max_{i \neq 0} |1 - \lambda_i| < \delta$. Suppose X is a subset of vertices of G and i is a non-negative value. We have*

$$\sum_{v \in X} \frac{1}{d_v^{i+1}} (\mathrm{vol}_{-i}(\Gamma_X(v)))^2 = \frac{\mathrm{vol}_{-i+1}^3(X)}{\mathrm{vol}^2(G)} + O(\delta \frac{\mathrm{vol}_{-i+1}(X) \mathrm{vol}_{-2i+1}(X)}{\mathrm{vol}(G)})$$

and

$$\begin{aligned} &\sum_{v \in X} \frac{1}{d_v^{i+1}} \mathrm{vol}_{-i}(\Gamma_X(v)) \mathrm{vol}_{-i}(R(v)) \\ &\qquad = \frac{\mathrm{vol}_{-i+1}(X)}{\mathrm{vol}(G)} \sum_{\{u,v\} red} \frac{1}{d_v^i d_u^i} + O(\bar{\lambda} \frac{\mathrm{vol}_{-i+1}(X) \mathrm{vol}_{-2i+1}(X)}{\mathrm{vol}(G)}). \end{aligned}$$

Proof:

$$\begin{aligned}\sum_{v\in X}\frac{1}{d_v^{i+1}}(\mathrm{vol}_{-i}(\Gamma_X(v)))^2 &= \frac{\mathrm{vol}^3_{-i+1}(X)}{\mathrm{vol}^2(G)}+\sum_{v\in X}\frac{\mathrm{vol}^2_{-i}(\Gamma_X(v))-(d_v\mathrm{vol}_{-i+1}(X)/\mathrm{vol}(G))^2}{d_v^{i+1}}\\ &= \frac{\mathrm{vol}^3_{-i+1}(X)}{\mathrm{vol}^2(G)}+O(\bar\lambda\frac{\mathrm{vol}_{-i+1}(X)\mathrm{vol}_{-2i+1}(X)}{\mathrm{vol}(G)}),\end{aligned}$$

$$\begin{aligned}\sum_{v\in X}\frac{1}{d_v^{i+1}}\mathrm{vol}_{-i}(R(v))\mathrm{vol}_{-i}(\Gamma_X(v)) &= \sum_{v\in X}\frac{1}{d_v^{i}}\mathrm{vol}_{-i}(R(v))\frac{\mathrm{vol}_{-i+1}(X)}{\mathrm{vol}(G)}\\ &\quad+\sum_{v\in X}\frac{1}{d_v^{i+1}}\mathrm{vol}_{-i}(R(v))(\mathrm{vol}_{-i}(\Gamma_X(v))-\frac{d_v\mathrm{vol}_{-i+1}(X)}{\mathrm{vol}(G)})\\ &= \frac{\mathrm{vol}_{-i+1}(X)}{\mathrm{vol}(G)}\sum_{\{u,v\}red}\frac{1}{d_v^i d_u^i}+O(\bar\lambda\frac{\mathrm{vol}_{-i+1}(X)\mathrm{vol}_{-2i+1}(X)}{\mathrm{vol}(G)}).\end{aligned}$$

□

Lemma 6 *Suppose that X is a subset of vertices in a graph G and $\alpha\le\beta$ are non-negative values. Then*

$$\mathrm{vol}_{-\alpha}(X)\mathrm{vol}_{-\beta}(X)\le\mathrm{vol}_{-\alpha+1}(X)\mathrm{vol}_{-\beta-1}(X). \tag{6}$$

Proof: The inequality (6) follows from the following general version of the Cauchy-Schwarz inequality for positive a_j's and $0\le\alpha\le\beta$:

$$(\sum_{j=1}^k a_j^\alpha)(\sum_{j=1}^k a_j^\beta)\le(\sum_{j=1}^k a_j^{\alpha-1})(\sum_{j=1}^k a_j^{\beta+1}).$$

By choosing a_j's to be the reciprocal of the degrees, (6) is an immediate consequence. □

4 A generalization of Turán's theorem

We will now prove the main theorem:

Theorem 1 *Suppose a graph G on n vertices has eigenvalues $0=\lambda_0\le\lambda_1\le\ldots\le\lambda_{n-1}$ with $\bar\lambda=\max_{i\neq0}|1-\lambda_i|$ satisfying*

$$\bar\lambda=o(\frac{1}{\mathrm{vol}_{-2t+3}(G)\mathrm{vol}(G)^{t-2}}). \tag{7}$$

Then, G is t-Turán for $t\ge2$; i.e., any subgraph of G containing no K_{t+1} has at most $(1-1/t+o(1))e(G)$ edges where $e(G)$ is the number of edges in G.

There are expressions in terms of $o(\cdot)$'s in the statement of Theorem 1. To be precise, the result in Theorem 1 can be restated as follows:

For any $\epsilon > 0$, there is a δ such that if the eigenvalues of the Laplacian of the graph G satisfies

$$\bar{\lambda} = \max_{i \neq 0} |1 - \lambda_i| < \frac{\delta}{\mathrm{vol}_{-2t+3}(G)\mathrm{vol}(G)^{t-2}},$$

then any subgraph of G containing no K_{t+1} has at most $(1 - 1/t + \epsilon)e(G)$ edges where $e(G)$ is the number of edges in G.

We remark that the condition in (7) has the following implication for the minimum degree η of G.

$$\begin{aligned} \bar{\lambda} &\ll \frac{1}{\mathrm{vol}_{-2t+3}(G)\mathrm{vol}(G)^{t-2}} \\ &< \frac{\eta^{2t-3}}{\mathrm{vol}(G)^{t-2}n} \leq \frac{\eta^{2t-3}}{n^{t-1}\eta^{t-2}} = \frac{\eta^{t-1}}{n^{t-1}} \end{aligned}$$

Since the inequality

$$\bar{\lambda} \geq 1/\sqrt{\eta n}$$

always holds, condition (7) implies that

$$\eta \gg n^{(2t-3)/(2t-1)}.$$

Thus, condition (7) may hold only if the minimum degree of the graph G is sufficiently large.

Theorem 1 implies the following two facts – the Classical Turán therem and the case for regular graphs,

Corollary 1 *A graph on n vertices containing no K_{t+1} has at most $(1 - 1/t + o(1))n^2/2$ edges.*

Proof: The complete graph on n vertices has Laplacian eigenvalues 0 and $n/(n-1)$ (with multiplicity $n-1$). Thus, for $G = K_n$, it is always true that

$$\bar{\lambda} = \frac{1}{n-1} = o(1).$$

Therefore, Theorem 1 implies that any graph on n vertices containing no K_{t+1} has at most $(1-1/t+o(1))n^2/2$ edges. □

Corollary 2 *If a graph is regular with degree d and has n vertices, then the condition in (7) is just*

$$\bar{\lambda} = o\left((\frac{d}{n})^{t-1} \right).$$

Then any subgraph of G containing no K_{t+1} has at most $(1 - 1/t + o(1))dn/2$ edges.

The proof is by directly substitution and will be omitted.

Suppose that R is a subset of edges so that every K_{t+1} in G contains at least one edge in R. In order to prove Theorem 1, we wish to show that $|R| \geq (1 + o(1))|E(G)|/t$.

To do so, we will prove the following stronger result:

Theorem 2 *Suppose a graph G on n vertices has eigenvalues satisfying (7). Then we have*

(*) For any subset X of vertices in G and for non-negative integer $k \leq t$, if R contains an edge from every complete subgraph on $k+1$ vertices, then we have, for all i, $0 \leq i \leq k$,

$$\sum_{v \in X, u \in X, \{u,v\} \in R} \frac{1}{d_u^i d_v^i} \geq \frac{\mathrm{vol}_{-i+1}^2(X)}{k\mathrm{vol}(G)} + O\left(\bar{\lambda} \sum_{j=0}^{k-1} \mathrm{vol}_{-2i-2j+1}(X)\mathrm{vol}^j(G)\right) \tag{8}$$

To derive Theorem 1 from Theorem 2, suppose that R contains an edge from every complete subgraph on $k+1$ vertices. We apply (*) with $i = 0$ and $k = t$. We then use (7) and have

$$\begin{aligned}
|R| &\geq \frac{\mathrm{vol}(G)}{t} + O(\bar{\lambda} \sum_{j=0}^{t-1} \mathrm{vol}_{-2j+1}(G)\mathrm{vol}^j(G)) \\
&\geq \frac{\mathrm{vol}(G)}{t}(1 + O(\bar{\lambda}\mathrm{vol}_{-2t+3}(G)\mathrm{vol}^{t-2}(G))) \\
&\geq (1 + o(1))\frac{\mathrm{vol}(G)}{t}.
\end{aligned}$$

Therefore G is t-Turán, as claimed. □

5 Proofs of the main theorems

In this section, we shall prove Theorem 2. The inequality in (*) is somewhat complicated. We shall deal with simpler cases first (such as $i = 0$) which contain the main ideas.

Proof of Theorem 2:

First we want to show that (*) holds for $k = 1$. In this case we have $R = E(G)$. Lemma 2 implies that (*) holds for $k = 1$.

Suppose that $k \geq 2$ and (*) holds for $k' < k$. We wish to prove (*) for k. Suppose R contains an edge from every complete subgraph on $k+1$ vertices. We want to show that the inequality (8) holds for all i, $0 \leq i \leq k$.

We shall first prove the case that $i = 0$. Suppose that edges in R are colored *red* and the rest of the edges of G are *blue*. We focus on edges inside of the given set X. Recall that $\Gamma_X(v) = \Gamma(v) \cap X$. For each vertex v, let $R(v)$ and $B(v)$ denote the set of neighbors u of v in X with $\{u, v\}$ red and blue, respectively. For a vertex v, we consider the induced subgraph on $B(v)$ which does not contain a complete graph on k vertices. By induction we have

$$\sum_{u,w\in B(v),\{u,w\}\text{red}} \frac{1}{d_u d_w} \geq \frac{1}{(k-1)} \frac{|B(v)|^2}{\mathrm{vol}(G)} + O\left(\bar{\lambda}\sum_{j=0}^{k-2} \mathrm{vol}_{-2j-1}(B(v))\mathrm{vol}^j(G)\right). \tag{9}$$

We consider the set T_j of all triangles in X containing exactly j red edges, for $j = 1, 2, 3$.

$$\begin{aligned}
W_1 &= \sum_{\{u,v,w\}\in T_1} \frac{2}{d_v d_u d_w} \\
&= \sum_{v\in X} \frac{1}{d_v} \sum_{u,w\in B(v),\{u,w\}\text{red}} \frac{1}{d_u d_w} \\
&\leq \sum_{v\in X} \frac{1}{d_v} \sum_{u\in R(v)} \sum_{w\in\Gamma_X(v)\cap\Gamma(u)} \frac{1}{d_u d_w} - \sum_{\{v,u,w\}\in T_2} \frac{2}{d_v d_u d_w} - \sum_{\{v,u,w\}\in T_3} \frac{3}{d_v d_u d_w} \\
&\leq \sum_{v\in X} \frac{1}{d_v} \sum_{u\in R(v)} \sum_{w\in\Gamma_X(v)\cap\Gamma(u)} \frac{1}{d_u d_w} - 2W_2 - 3W_3
\end{aligned}$$

We note that

$$W_2 + 3W_3 = \sum_{v\in X} \frac{1}{d_v} \sum_{u,w\in R(v),u\sim w} \frac{1}{d_u d_w}.$$

Thus,

$$\begin{aligned}
W_1 &\leq \sum_{v\in X} \frac{1}{d_v} \sum_{u\in R(v)} \sum_{w\in\Gamma_X(v),w\sim u} \frac{1}{d_u d_w} - W_2 - 3W_3 \\
&\leq \sum_{v\in X} \frac{1}{d_v} \sum_{u\in R(v)} \sum_{w\in\Gamma_X(v),w\sim u} \frac{1}{d_u d_w} - \sum_{v\in X} \frac{1}{d_v} \sum_{u,w\in R(v),u\sim w} \frac{1}{d_u d_w}.
\end{aligned}$$

By Lemma 2, we have

$$\begin{aligned}
\sum_{u\in R(v)} \sum_{w\in\Gamma_X(v),w\sim u} \frac{1}{d_u d_w} &= \frac{|R(v)|\ |\Gamma_X(v)|}{\mathrm{vol}(G)} + O(\bar{\lambda}\sqrt{\mathrm{vol}_{-1}R(v)\mathrm{vol}_{-1}(\Gamma_X(v))}) \\
\sum_{u,w\in R(v),u\sim w} \frac{1}{d_u d_w} &= \frac{|R(v)|^2}{\mathrm{vol}(G)} + O(\bar{\lambda}\mathrm{vol}_{-1}(R(v)).
\end{aligned}$$

Therefore,

$$W_1 \leq \sum_{v\in X} \frac{1}{d_v} \frac{|\Gamma_X(v)|\ |R(v)|}{\mathrm{vol}(G)} - \sum_{v\in X} \frac{1}{d_v} \frac{|R(v)|^2}{\mathrm{vol}(G)} + O(\bar{\lambda} \sum_{v\in X} \frac{\mathrm{vol}_{-1}(\Gamma_X(v))}{d_v}). \tag{10}$$

On the other hand, from (10) we have

$$\begin{aligned}
W_1 &= \sum_{\{u,v,w\}\in T_1} \frac{2}{d_v d_u d_w} \\
&\geq \frac{1}{(k-1)} \sum_{v\in X} \frac{1}{d_v} \frac{|B(v)|^2}{\mathrm{vol}(G)} - O(\bar{\lambda} \sum_{j=0}^{k-2} \frac{\mathrm{vol}_{-2j-1}(B(v))\mathrm{vol}^j(G)}{d_v}) \\
&\geq \frac{1}{(k-1)} \sum_{v\in X} \frac{1}{d_v} \frac{|B(v)|^2}{\mathrm{vol}(G)} - O(\bar{\lambda} \sum_{j=0}^{k-2} \frac{\mathrm{vol}_{-2j-1}(\Gamma_X(v))\mathrm{vol}^j(G)}{d_v}) \\
&\geq \frac{1}{(k-1)} \sum_{v\in X} \frac{1}{d_v} \frac{(|\Gamma_X(v)| - |R(v)|)^2}{\mathrm{vol}(G)} - O(\bar{\lambda} \sum_{j=0}^{k-2} \frac{|X|\mathrm{vol}_{-2j}(X)\mathrm{vol}^j(G)}{\mathrm{vol}(G)}) \\
&\quad + O(\bar{\lambda}^2 \sum_{j=0}^{k-2} \sqrt{\mathrm{vol}_{-1}(X)\mathrm{vol}_{-4j-1}(X)}\mathrm{vol}^j(G)).
\end{aligned}$$

We note that the terms involving $\bar{\lambda}^2$ are of lower order by using the assumption on $\bar{\lambda}$. Combining the preceding upper and lower bounds for W_1, we have

$$(k+1) \sum_{v\in X} \frac{|\Gamma_X(v)| \ \ |R(v)|}{d_v} \geq \sum_{v\in X} \frac{|\Gamma_X(v)|^2}{d_v} + k \sum_{v\in X} \frac{|R(v)|^2}{d_v} - O\left(\bar{\lambda}|X| \sum_{j=0}^{k-2} \mathrm{vol}_{-2j}(X)\mathrm{vol}^j(G) \right). \tag{11}$$

By Lemma 4 and inequality (5), we have

$$\sum_{v\in X} \frac{|\Gamma_X(v)| \ \ |R(v)|}{d_v} \leq \sum_{v\in X} \frac{|R(v)|\mathrm{vol}(X)}{\mathrm{vol}(G)} + O(\bar{\lambda}\frac{\mathrm{vol}(X)^2}{\mathrm{vol}(G)}) + O(\bar{\lambda}^2 \mathrm{vol}(X))$$

Also, by Lemma 4 and inequality (4), we have

$$\sum_{v\in X} \frac{|\Gamma_X(v)|^2}{d_v} \geq \frac{\mathrm{vol}^3(X)}{\mathrm{vol}^2(G)} + O(\bar{\lambda}\frac{\mathrm{vol}(X)^2}{\mathrm{vol}(G)}) + O(\bar{\lambda}^2 \mathrm{vol}(X)).$$

Substituting into (11), we have

$$\begin{aligned}
(k+1) \sum_{v} |R(v)| \frac{\mathrm{vol}(X)}{\mathrm{vol}(G)} &\geq \frac{\mathrm{vol}^3(X)}{\mathrm{vol}^2(G)} + k \sum_{v\in X} \frac{|R(v)|^2}{d_v} + O\left(\bar{\lambda}(\frac{\mathrm{vol}(X)^2}{\mathrm{vol}(G)} + \sum_{j=0}^{k-2} |X|\mathrm{vol}_{-2j}(X)\mathrm{vol}^j(G)) \right) \\
&\geq \frac{\mathrm{vol}^3(X)}{\mathrm{vol}^2(G)} + k \frac{(\sum_{v\in X} |R(v)|)^2}{\sum_{v\in X} d_v} + O\left(\bar{\lambda}(\frac{\mathrm{vol}(X)^2}{\mathrm{vol}(G)} + \sum_{j=0}^{k-2} |X|\mathrm{vol}_{-2j}(X)\mathrm{vol}^j(G)) \right).
\end{aligned}$$

This implies

$$(\frac{\mathrm{vol}^2(X)}{\mathrm{vol}(G)} - \sum_{v} |R(v)|)(\frac{\mathrm{vol}^2(X)}{\mathrm{vol}(G)} - k \sum |R(v)|) + O\left(\bar{\lambda}(\frac{\mathrm{vol}(X)^3}{\mathrm{vol}(G)} + \sum_{j=0}^{k-2} |X|\mathrm{vol}_{-2j}(X)\mathrm{vol}^j(G)\mathrm{vol}(X) \right) < 0.$$

Thus we have

$$
\begin{aligned}
|R| &= \frac{1}{2}\sum_v |R(v)| \\
&\geq \frac{1}{2k}\frac{\mathrm{vol}^2(X)}{\mathrm{vol}(G)} + O\left(\bar{\lambda}(\mathrm{vol}(X) + \sum_{j=0}^{k-2}\frac{|X|\mathrm{vol}_{-2j}(X)\mathrm{vol}^{j+1}(G)}{\mathrm{vol}(X)})\right) \\
&\geq \frac{1}{2k}\frac{\mathrm{vol}^2(X)}{\mathrm{vol}(G)} + O\left(\bar{\lambda}(\mathrm{vol}(X) + \sum_{j=0}^{k-2}\mathrm{vol}_{-2j-1}(X)\mathrm{vol}^{j+1}(G))\right) \\
&\geq \frac{1}{k}|E(X)| + O\left(\bar{\lambda}(\mathrm{vol}(X) + \sum_{j=1}^{k-1}\mathrm{vol}_{-2j+1}(X)\mathrm{vol}^{j}(G))\right) \\
&\geq \frac{1}{k}|E(X)| + O\left(\bar{\lambda}\sum_{j=0}^{k-1}\mathrm{vol}_{-2j+1}(X)\mathrm{vol}^{j}(G)\right).
\end{aligned}
$$

We have completed the proof for the case $i = 0$.

Suppose $i \geq 1$. For $j = 1, 2, 3$, we consider

$$
W_j^{(i)} = \sum_{\{u,v,w\}\in T_j} \frac{2}{d_v^i d_u^i d_w^i}.
$$

As before, we have

$$
\begin{aligned}
& W_1^{(i+1)} \\
\leq\; & \sum_{v\in X}\frac{1}{d_v^{i+1}}\sum_{u\in R(v)}\sum_{w\in\Gamma_X(v)\cap\Gamma(u)}\frac{1}{d_u^{i+1}d_w^{i+1}} - W_2^{(i+1)} - 3W_3^{(i+1)} \\
\leq\; & \sum_{v\in X}\frac{1}{d_v^{i+1}}\sum_{u\in R(v)}\sum_{w\in\Gamma_X(v)\cap\Gamma(u)}\frac{1}{d_u^{i+1}d_w^{i+1}} - \sum_{v\in X}\frac{1}{d_v^{i+1}}\sum_{u,w\in R(v),u\sim w}\frac{1}{d_u^{i+1}d_w^{i+1}} \\
\leq\; & \sum_{v\in X}\frac{1}{d_v^{i+1}}\frac{\mathrm{vol}_{-i}(R(v))\mathrm{vol}_{-i}(\Gamma_X(v))}{\mathrm{vol}(G)} - \sum_{v\in X}\frac{1}{d_v^{i+1}}\frac{\mathrm{vol}_{-i}^2(R(v))}{\mathrm{vol}(G)} + O(\bar{\lambda}\sum_{v\in X}\frac{\mathrm{vol}_{-2i-1}(\Gamma_X(v))}{d_v^{i+1}}) \\
\leq\; & \sum_{v\in X}\frac{1}{d_v^{i+1}}\frac{\mathrm{vol}_{-i}(R(v))\mathrm{vol}_{-i}(\Gamma_X(v))}{\mathrm{vol}(G)} - \sum_{v\in X}\frac{1}{d_v^{i+1}}\frac{\mathrm{vol}_{-i}^2(R(v))}{\mathrm{vol}(G)} \\
& + O(\bar{\lambda}\frac{\mathrm{vol}_{-i}(X)\mathrm{vol}_{-2i}(X)}{\mathrm{vol}(G)} + \bar{\lambda}^2(\mathrm{vol}_{-2i-1}(X)\mathrm{vol}_{-4i-1}(X))^{1/2}).
\end{aligned}
$$

On the other hand, by induction we have

$$
\begin{aligned}
W_1^{(i+1)} &= \sum_{\{u,v,w\}\in T_1} \frac{2}{d_v^{(i+1)} d_u^{(i+1)} d_w^{(i+1)}} \\
&\geq \frac{1}{(k-1)} \sum_{v\in X} \frac{1}{d_v^{(i+1)}} \left(\frac{\mathrm{vol}_{-i}^2(B(v))}{\mathrm{vol}(G)} + O(\bar{\lambda} \sum_{j=0}^{k-2} \mathrm{vol}_{-2i-2j+1}(B(v)) \mathrm{vol}^j(G) \right) \\
&\geq \frac{1}{(k-1)} \sum_{v\in X} \frac{1}{d_v^{i+1}} \frac{\mathrm{vol}_{-i}^2(\Gamma_X(v)) - \mathrm{vol}_{-i}(R(v))}{\mathrm{vol}(G)} \\
&\quad + O\left(\bar{\lambda} \sum_{j=0}^{k-2} \frac{\mathrm{vol}_{-i}(X)\mathrm{vol}_{-2i-2j}(X)\mathrm{vol}^j(G)}{\mathrm{vol}(G)} + \bar{\lambda}^2 \sum_{j=0}^{k-2} \sqrt{\mathrm{vol}_{-2i-1}(X)\mathrm{vol}_{-4i-4j-1}(X)} \mathrm{vol}^j(G) \right).
\end{aligned}
$$

By combining the upper and lower bounds of $W_1^{(i+1)}$ (and multiplying by $\mathrm{vol}(G)$), we have

$$
(k+1)A + B \leq kC + O(\bar{\lambda} D \mathrm{vol}(G)) \tag{12}
$$

where A, B, C, D are as follows:

$$
\begin{aligned}
A &= \sum_{v\in X} \frac{1}{d_v^{i+1}} (\mathrm{vol}_{-i}(\Gamma_X(v)))^2 \\
&\geq \frac{\mathrm{vol}_{-i+1}^3(X)}{\mathrm{vol}^2(G)} + O(\bar{\lambda} \frac{\mathrm{vol}_{-i+1}(X)\mathrm{vol}_{-2i+1}(\Gamma(v))}{\mathrm{vol}(G)}) \qquad \text{by using Lemma 5.} \\
B &= \sum_{v\in X} \frac{1}{d_v^{i+1}} \mathrm{vol}_{-i}^2(R(v)) \\
&\geq (\sum_{v\in X} \frac{\mathrm{vol}_{-i}(R(v))}{d_v^i})^2 / \mathrm{vol}_{-i+1}(G) \qquad \text{by the Cauchy-Schwarz inequality.} \\
C &= \sum_{v\in X} \frac{1}{d_v^{i+1}} \mathrm{vol}_{-i}(\Gamma_X(v)) \mathrm{vol}_{-i}(R(v)) \\
&\leq \frac{\mathrm{vol}_{-i+1}(X)}{\mathrm{vol}(G)} \sum_{\{u,v\}\mathrm{red}} \frac{1}{d_v^i d_u^i} + O(\bar{\lambda} \frac{\mathrm{vol}_{-i+1}(X)\mathrm{vol}_{-2i+1}(X)}{\mathrm{vol}(G)}) \\
D &= \sum_{j=0}^{k-2} \mathrm{vol}_{-i}(X)\mathrm{vol}_{-2i-2j}(X)\mathrm{vol}^j(G) + \mathrm{vol}_{-i+1}(X)\mathrm{vol}_{-2i+1}(X)
\end{aligned}
$$

By substituting A, B and C into (12), we have the following:

$$
k\frac{\mathrm{vol}_{-i+1}^3(X)}{\mathrm{vol}^2(G)} + \frac{(\sum_{\{u,v\}\mathrm{red}} \frac{1}{d_v^i d_u^i})^2}{\mathrm{vol}_{-i+1}(X)} \leq (k+1)\frac{\mathrm{vol}_{-i+1}(X)}{\mathrm{vol}(G)} \sum_{\{u,v\}\mathrm{red}} \frac{1}{d_v^i d_u^i} + O\left(\bar{\lambda} D \mathrm{vol}(G)\right).
$$

This implies

$$
\sum_{\{u,v\}\mathrm{red}} \frac{1}{d_v^i d_u^i} \quad \geq \quad \frac{\mathrm{vol}_{-i+1}^2(X)}{k\mathrm{vol}(G)} - O\left(\bar{\lambda}(\mathrm{vol}_{-2i+1}(X) + \sum_{j=0}^{k-2} \frac{\mathrm{vol}_{-i}(X)\mathrm{vol}_{-2i-2j}(X)\mathrm{vol}^{j+1}(G)}{\mathrm{vol}_{-i+1}(X)}) \right).
$$

Now, by using Lemma 6 and inequality (6), we have

$$\begin{aligned}\sum_{\{u,v\}\text{red}} \frac{1}{d_v^i d_u^i} &\geq \frac{\text{vol}_{-i+1}^2(X)}{k\text{vol}(G)} + O(\bar{\lambda}\text{vol}_{-2i+1}(X) + \sum_{j=0}^{k-2} \text{vol}_{-2i-2j-1}(X)\text{vol}^{j+1}(G)) \\ &\geq \frac{\text{vol}_{-i+1}^2(X)}{k\text{vol}(G)} + O(\bar{\lambda}\sum_{j=0}^{k-1} \text{vol}_{-2i-2j+1}(X)\text{vol}^{j}(G)).\end{aligned}$$

This completes the proof of Theorem 2. □

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